MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS;

FROM "THE EDUCATIONAL TIMES."

VOL. II.

MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

WITH MANY

Additional Solutions not published in the "Educational Times."

EDITED BY

W. J. MILLER, B.A.,

MATHEMATICAL MASTER, HUDDERSFIELD COLLEGE.

VOL. II.

FROM JULY TO DECEMBER, 1864.

LONDON:

C. F. HODGSON & SON, GOUGH SQUARE,

FLEET STREET.

1865.

LONDON:
PRINTED BY C. F. HODGSON & SON,
GOUGH SQUARE, FLEET STREET.

LIST OF CONTRIBUTORS

TO

VOLUME II.

ALLEN, J. R., Rugby School.
ANDERSON, D. M., Kirriemuir, Scotland.
BILLS, S., Hawton, Newark-on-Trent.
BLISSARD, Rev. J., B. A., The Vicarage, Hampstead Norris, Berks.
BOOTH, Rev. Dr., F.R.S., The Vicarage, Stone, Bucks.
CASEY, J., B.A., Kingstown, Ireland.
CAYLEY, Professor, F.R.S., University of Cambridge; Corresponding Member of the Institute of France.
CLIFFORD, W. K., Trinity College, Cambridge.
COCKLE, the Hon. J., M.A., Chief Justice of Queensland.
COLLINS, MATTHEW, B.A., Dublin.
CONWILL, J., Leighlinbridge, Ireland.
CREMONA, Professor, University of Bologna.
CROFTON, M. W., B.A., Royal Military Academy, Woolwich.
DOBSON, T., B.A., Head Master of Hexham Grammar School.
EASTERBY, W., B.A., Grammar School, St. Asaph.
FENWICK, STEPHEN, F.R.A.S., Royal Military Academy, Woolwich.
FLOOD, P. W., Ballingarry, Ireland.
GODFRAY, HUGH, M.A., Cambridge.
GODWARD, W., Law Life Office, London.
GREER, H. R., B.A., Royal Military College, Sandhurst.
GRIFFITHS, J., M.A., Jesus College, Oxford.
HARLEY, Rev. R., F.R.S., Brighouse, Yorkshire.
HIRST, Dr. T. A., F.R.S., London.
HOLDITCH, Rev. H., Caius College, Cambridge.
HOPPS, W., Hull.

HUDSON, W. H. H., M.A., St. John's College, Cambridge.
KIRKMAN, Rev. T. P., M.A., F.R.S., Croft Rectory, near Warrington.
LEVY, W. H., Shalbourne, Berks.
McDOWELL, J., M.A., F.R.A.S., Pembroke College, Cambridge.
MILLER, W. J., B.A., Huddersfield College.
MURPHY, H., Killeshandra, Ireland.
NELSON, R. J., M.A., London.
O'CALLAGHAN, J., Cahir, Ireland.
PALMER, R., M.A., London.
PHILLIPS, W. O., M.A.
POOLEY, T.
PURKISS, H. J., B.A., Trinity College, Cambridge.
RENSHAW, A., Sherwood Rise, Nottingham.
RUTHERFORD, Dr., F.R.A.S., Woolwich.
SADLER, G. T., F.R.A.S., London.
SALMON, Rev. Dr., F.R.S., Trinity College, Dublin.
SPOTTISWOODE, W., F.R.S., London.
STANLEY, ARCHER.
SYLVESTER, Professor, F.R.S., Royal Military Academy, Woolwich; Corresponding Member of the Institute of France.
TAYLOR, C., M.A., St. John's College, Cambridge.
THOMSON, F. D., M.A., Leamington College.
TOWNSEND, Rev. R., M.A., Trinity College, Dublin.
TUCKER, R., M.A., Newport, Isle of Wight.
WATSON, STEPHEN, Haydonbridge.
WHITWORTH, W. A., B.A., Runcorn.
WILKINSON, T. T., F.R.A.S., Burnley.
WILSON, J. M., M.A., Rugby School.
WILSON, J., 45th Regiment, Colaba Camp, Bombay.
WOOLHOUSE, W. S. B., F.R.A.S., &c., London.
WRIGHT, Rev. R. H., M.A., Head Master of Ashford Grammar School.

CONTENTS.

MATHEMATICAL QUESTIONS

WITH THEIR

Solutions.

FROM

THE EDUCATIONAL TIMES.

1492 (Proposed by Dr. BOOTH, F.R.S.)—Show that the tangential equation of the caustic by reflexion of the circle is

$$\left[(2\rho)^2-(\xi+\gamma)^2\right](\xi^2+v^2)=4\rho^2(\rho^2-\gamma\xi),$$

ξ and v being the tangential coordinates of the curve, while ρ and γ are the reciprocals of the radius of the given circle, and of the distance of the radiant point from its centre.

When the incident rays are parallel, $\gamma=0$, and the equation becomes

$$(4\rho^2-\xi^2)\,v^2=(2\rho^2-\xi^2)^2.$$

I. *Solution by the* PROPOSER.

Let the equation of the circle be

$$x^2+y^2=4a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and let the reciprocal of the distance of the radiant point, taken in the axis of x, from the centre be γ. Then, as the angles of incidence and reflexion are equal, and as the reflected ray is a tangent to the caustic, the sides of this triangle will be in the same ratio as the segments of the base, made by the radius passing through the point of incidence. Hence

$$\left\{y^2+\left(x-\frac{1}{\gamma}\right)^2\right\} : \left\{y^2+\left(x-\frac{1}{\xi}\right)^2\right\} = \frac{1}{\gamma^2} : \frac{1}{\xi^2},$$

or reducing, and introducing the relation $x^2+y^2=4a^2$, we shall have

$$x=2a^2(\xi+\gamma) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

But the general relation between projective and tangential coordinates being

$$x\xi + yv = 1 \ldots\ldots\ldots\ldots (3),$$

we shall have, substituting these values of x and y in (1),

$$4a^2 [1-a^2 (\xi+\gamma)^2] (\xi^2+v^2) = 1-4a^2\gamma\xi \ldots\ldots (4)$$

the general tangential equation of the caustic by reflexion of the circle.

If we put $\rho = \dfrac{1}{2a}$, the reciprocal of the radius, we obtain the equation in the form given in the question.

Discussion.—(i.) Let $\xi = -\gamma$, then $2av = \pm 1$, or if a line be drawn from either end of the vertical diameter of the circle to a point on the axis of x, on the opposite side to the radiant point and equidistant from the centre, this line will always be a tangent to the caustic.

(ii.) To find the limits of the curve between the vertical tangents to it. When the tangent is vertical, $v = 0$, and the general equation becomes

$$[1-2a^2\xi(\xi+\gamma)]^2 = 0,$$

$$\text{whence } \xi = -\frac{\gamma}{2} \pm \frac{\sqrt{(2+a^2\gamma^2)}}{2a}.$$

(iii.) When $\xi = 0$, $v = \dfrac{\pm 1}{2a\sqrt{(1-a^2\gamma^2)}}$, or the tangent to the caustic parallel to the axis of x cuts the axis of y at the distances $\pm 2a\sqrt{(1-a^2\gamma^2)}$ from the origin.

(iv.) When $y = 0$, $v = \infty$, and as the second side of the general equation (4) is finite, we must have the factor $[1-a^2(\xi+\gamma)^2] = 0$, or $\xi+\gamma = \dfrac{1}{a}$, which is the general equation between the conjugate foci in optics.

(v.) If from the radiant point we draw a tangent to the caustic, it will also be a tangent to the circle. For if in the general equation we put γ for ξ, we shall find $v = \dfrac{\sqrt{(1-4a^2\gamma^2)}}{2a}$, and this may easily be shown to be the reciprocal of the segment cut off from the axis of y by the tangent to the circle from the radiant point.

It is needless to pursue this investigation further. The power of the method is evident.

Note.—The above demonstration I have copied from one of my old college note-books, dated 22nd August, 1837. I was led to its investigation from the following remark made by Gergonne, in the *Annales de Mathématiques*, tom. xv. page 346:—"J'étais depuis quelque temps en possession de l'équation de la caustique par reflexion sur le cercle, qui n'avait encore été donnée par personne; mais je l'avais obtenu par des calculs trop prolixes, et sous une forme trop peu élégante pour songer à la publier," &c.

In 1840, shortly after taking my Degree, I printed a little tract on *Tangential Coordinates*, in which most of the elementary transformations of the correlative formulæ, their applications to the Conic Sections and many well-known curves are given. In the same publication will be found many of the analogies between the reciprocal surfaces

$$F(x, y, z, a, b, c, \&c.) = 0 \quad \text{and} \quad \Phi(\xi, v, \zeta, \alpha, \beta, \gamma, \&c.) = 0.$$

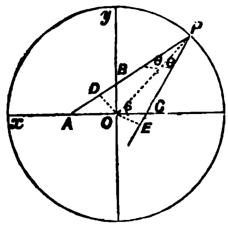
The discovery of this method attracted but little notice at the time, although

it was noticed shortly after publication by the late D. F. Gregory, of Cambridge, at p. 226 of his *Examples on the Processes of the Differential and Integral Calculus*, published in 1841:—" He will also find these questions, along with others of a similar kind, very ingeniously treated in a short tract on 'Tangential Coordinates,' by J. Booth, of Trinity College, Dublin. The method employed by that author does not come within the scope of the present work, but it merits attention, as affording a ready solution of many curious problems which yield with difficulty to the power of ordinary analysis."

It is irksome to be forced to refer to these facts, but as the method of Tangential Coordinates would seem now to be gradually creeping into use, it becomes proper that I should assert my claim to be the inventor of this most powerful instrument of mathematical investigation, the necessary complement to the projective coordinates of Descartes; the more so, that it has been ignored by some of our mathematical book-makers.

[Dr. Booth's system of Tangential Coordinates has been also noticed in Salmon's *Higher Curves*, Arts. 13 and 16 (Note); in Walton's *Problems in Coordinate Geometry*, p. 416, Exs. 2, 5, 6, 10; in Ferrers's *Trilinear Coordinates*, p. 131; and in the *Quarterly Journal of Mathematics*, Vol. ii., p. 221.—EDITOR.]

II. *Solution by* F. D. THOMSON, B.A.



Let C be the radiant point, CPA the course of a ray, meeting the diameter COA and the perpendicular diameter in A and B respectively.

Put $OA = \dfrac{1}{u}$, $OB = \dfrac{1}{v}$, $OC = \dfrac{1}{c}$, $OP = \dfrac{1}{\gamma}$,

$$\angle POC = \phi, \quad \angle CPO = OPA = \theta.$$

Draw OD, OE perpendicular to PA, PC respectively, then

$$OD = OE = \frac{\sin\theta}{r},$$

$$OA = \frac{OD}{\sin(\phi-\theta)} = \frac{\sin\theta}{r\sin(\phi-\theta)}, \quad OB = \frac{OD}{\cos(\phi-\theta)} = \frac{\sin\theta}{r\cos(\phi-\theta)},$$

$$OC = \frac{OD}{\sin(\phi+\theta)} = \frac{\sin\theta}{r\sin(\phi+\theta)}.$$

$$\therefore \quad u\sin\theta = r\sin(\phi-\theta) \ldots\ldots\ldots\ldots \text{(i.)}$$
$$v\sin\theta = r\cos(\phi-\theta) \ldots\ldots\ldots\ldots \text{(ii.)}$$
$$c\sin\theta = r\sin(\phi+\theta) \ldots\ldots\ldots\ldots \text{(iii.)}$$

From (i.) and (ii.), $(u^2+v^2)\sin^2\theta = r^2 \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$

From (i.) and (iii.), $(c+u)\tan\theta = 2r\sin\phi$, and $(c-u) = 2r\cos\phi$;

$\therefore \; (c+u)^2\tan^2\theta + (c-u)^2 = 4r^2$, or $(c+u)^2 = 4(r^2+cu)\cos^2\theta \ldots\ldots\ldots (\beta).$

Eliminating θ from (α) and (β), we obtain

$$(u^2+v^2)\left\{4r^2-(c-u)^2\right\} = 4r^2(r^2+cu) \ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

If C be at an infinite distance, $c=0$, and the equation becomes

$$(u^2+v^2)(4r^2-u^2) = 4r^4, \text{ or } (4r^2-u^2)v^2 = (2r^2-u^2)^2 \ldots\ldots\ldots\ldots (\delta).$$

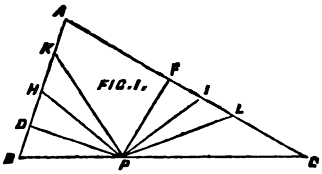
Equation (γ) agrees with that of Dr. Booth, if OC be reckoned *negative*.

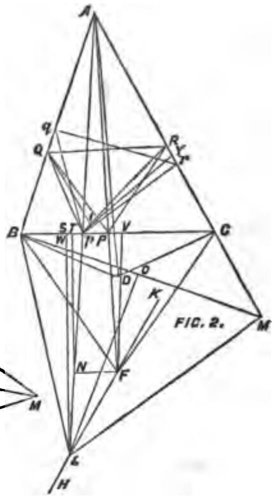
1428 (Proposed by the Editor.)—From a point in the base of a triangle two straight lines are drawn containing a given angle, and forming a quadrilateral with the other two sides of the triangle. Show that

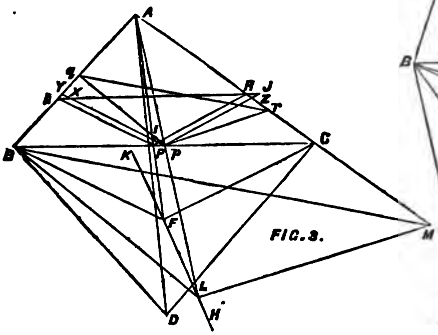
(α) Of all the quadrilaterals which have a common vertex at the *same* point within the vertical angle of the triangle, that of which the sides passing through this point are equal to each other is a *maximum*, a *minimum*, or *neither* a maximum nor a minimum, according as the given angle is *less* than, *greater* than, or *equal* to, the supplement of the vertical angle of the triangle.

(β) Of all these maximum or minimum quadrilaterals which have their vertices at *different* points in the base, the *least maximum* or the *greatest minimum* is that whose equal sides make equal angles with the base.

Solution by Mr. W. Hopps.







1. Let AHPI, AKPL (Fig. 1) be quaqrilaterals formed by drawing lines from a point P in the base BC of the triangle ABC meeting the other sides, PH being equal to PI and $\angle$ HPI = KPL. Draw PD, PF perpendicular to AB, AC. Obviously $\angle$ HPK = IPL, also $\angle$ DPF = the supplement of A, and when $\angle$ HPI < DPF, then $\angle$ DPH > FPI, therefore $\angle$ KHP (= PDH + DPH) > PIL (= PFI + FPI); hence the triangles PHK, PIL have PH = PI, $\angle$ HPK = IPL and $\angle$ KHP > PIL, therefore $\triangle$PHK > PIL, consequently AHPI > AKPL, that is, AHPI is a *maximum*.

2. When $\angle$ HPI = DPF, then $\angle$ DPH = FPI, and (Euc. i. 26) the triangles PDH, PFI are equal in all respects, hence PD = PF, consequently P is in the bisector of the angle A, and is therefore a fixed point. Moreover, $\angle$ KHP = PIL, hence the triangles PHK, PIL are equal, therefore AHPI = AKPL; that is, AHPI is neither a maximum nor a minimum.

3. When $\angle$HPI > DPF, then $\angle$DPH < FPI, therefore $\angle$KHP < PIL, hence $\triangle$PHK < PIL, consequently AHPI < AKPL, that is, AHPI is a *minimum*. And similar proofs obtain whatever may be the relative positions of the angles DPF, HPI, KPL; the theorems in (α) are therefore established.

4. Let ABC be a triangle (fig. 2 or 3) having AB < AC, then $\angle$ACB < ABC, and is therefore acute. Externally on the base BC construct the isosceles triangles DBC, FBC, the former having the vertical $\angle$BDC = the supplement of A, and the latter the vertical $\angle$BFC < BDC (fig. 2), and > BDC (fig. 3); through F draw the indefinite line HK making $\angle$BFK = ACB, and draw AF, AD, the former cutting BC in P. The points A, B, D, C are obviously in the circumference of a circle, and F is a point without this circle (fig. 2) and within it (fig. 3), therefore $\angle$AFB < ADB (fig. 2) and > ADB (fig. 3), hence $\angle$BFH + AFB < (BFH + ADB) (fig. 2) and > (BFH + ADB) (fig. 3), but $\angle$BFH = the supplement of BFK or ACB, and $\angle$ADB = ACB, consequently $\angle$BFH + AFB < two right $\angle$s (fig. 2) and > two right $\angle$s (fig. 3). Whence it is clear that if a point p be taken in BP (fig. 2) or PC (fig. 3), the line through A, p will meet FH at a point L. Join BL (figs. 2 and 3); through B draw a line meeting AC *externally* at M, making $\angle$CBM = FBL, and join L, M; then obviously FBL, CBM are similar triangles, therefore BL : BM = BF : BC, but $\angle$LBM = FBC, therefore LBM, FBC are also similar triangles, hence LM = LB. Draw PQ, PR parallel to FB, FC and pq, pr parallel to LB, LM, and join QR, pr; then obviously PQ = PR, $pq = pr$, QR is parallel to BC, $\angle$QPR = BFC, $\angle qpr$ = BLM = BFC and $\angle$QPB = PQR = PRQ = RPC, and we have to show that AQPR < Aqpr (fig. 2) and > Aqpr (fig. 3). Join pQ, pR (fig. 2); then as $\angle$QPB = QRP, it is plain (Euc. iii, 32) that BC is a tangent at P to the circle passing through PQR, and that this circle cuts pR at a point I between p and R. Join QI, then $\angle$QpR < QIR = QPR = qpr, i.e., $\angle$QpR < qpr. Through p draw a line making with Qp an angle equal to qpr, then it is clear that this line will meet AC at a point Y between R and r; therefore AQPR (= AQvR) < AQpY, which (by Art. i.) < Aqpr. Hence AQPR is, in this case, the *least maximum* quadrilateral.

5. Again, draw lines from p (fig. 3.) parallel to PQ, PR, the former meeting QR, AB at X, Y and the latter AC, QR at Z, J, and let I be the intersection of pY, PR; then QPpY > QPpX (= RPpJ) > RPpZ, therefore QPIY > RIpZ, consequently AQPR > AYpZ which (by Art. 3) > Aqpr, that is AQPR is the *greatest minimum* quadrilateral; and thus are proved the theorems in (β).

6. Through F (fig. 2) draw a line parallel to BC meeting AL at N; also draw LS, NT, FV each perpendicular to BC, and LO perpendicular to BM, and let W be the intersection of LS, BM; then evidently LS > LW > LO > FV or NT, hence Lp > Np, that is, FN cuts AL *internally* between the points p and L. Now Aq : AB = Ap : AL < Ap : AN = AP : AF = AQ : AB, i.e., Aq : AB < AQ : AB, therefore Aq < AQ, also Bp < BP, and $\angle qp$R < QpR < qpr, hence Cr < CR. Again, in fig. 3 it may be similarly proved that Aq < AQ and Cr < CR, but Bp > BP. When however p (figs. 2 and 3) is taken on the opposite side of P, the line passing through A, p will manifestly meet FK instead of FH at L, and in this case we must draw a line through B meeting AC *internally* at M, making $\angle$CBM = FBL; and then, proceeding as before, we shall find Aq > AQ, Bp > BP, and Cr > CR (Fig. 2), also that Aq > AQ, Bp > BP, and Cr > CR (Fig. 3). Thus are clearly exhibited the relative positions of the angular points P, Q, R and p, q, r of the quadrilaterals AQPR, Aqpr, according as the angle QPR (or qpr) is less or greater than the supplement of A.

7. **In arts. 4 and 5 are included solutions of the following propositions :—**

(*a*). In a given triangle to inscribe another of given species, one of whose angular points shall be coincident with a given point in one of the sides of the given triangle.

(*b*). A point and a straight line are given in position, and two angular points of a triangle given in species, are coincident with the given point and *any* point in the given line; then the locus of the remaining angular point of the triangle is a straight line given in position.

(*c*). To find a point in a side of a given triangle, such that if lines be drawn therefrom parallel to two lines given in position, meeting the other sides of the triangle, the quadrilateral thus formed shall be a maximum.

For, in figs. 2 and 3, the triangle *pqr*, which is given in species, is inscribed in the triangle ABC, so that a given angular point thereof is coincident with the given point *p* in the side BC.

Again, B is a point and AM a straight line given in position, and two of the angular points of the triangle LBM, which is given in species, are coincident with B and a point M in AM, and the indefinite straight line HK, which is manifestly given in position, is obviously the locus of the remaining angular point L of the triangle LBM. (See also McDowell's *Exercises*, Prop. 104; and Salmon's *Conics*, p. 53, Ex. 2.)

Also, P is a point (fig. 3) determined in the side BC of the triangle ABC, from which lines PQ, PR are drawn parallel to BF, CF, which may be parallel to *any* two lines given in position, and *p*Y, *p*Z are lines drawn from another point *p* in BC parallel to BF, CF, and, by the method of proof used in art. 5, we have AQPR > AY*p*Z, therefore AQPR is a maximum. This solution will be found much more simple than that given by SWALE in the 2nd number of the *Apollonius*, pp. 69—73.

In the solutions of (*a*) and (*b*), contained in Arts. 4 and 5, certain triangles are *isosceles*, but the method is clearly the same of *whatever species* these triangles may be.

1505 (Proposed by Professor CAYLEY.)—If P, Q, 1, 2, 3, 4 be points on a conic, then the four points P1, Q2; P2, Q1; P3, Q4; P4, Q3 lie on a conic passing through the points P and Q.

I. *Solution by the* PROPOSER.

This is an immediate consequence of the theorem of the anharmonic property of the points of a conic. For if (P1, P2, P3, P4) denote the anharmonic ratio of the lines P1, P2, P3, P4, and so in other cases; then

$$(P2, P1, P4, P3) = (P1, P2, P3, P4) = (Q1, Q2, Q3, Q4); \text{ that is,}$$

$$(P2, P1, P4, P3) = (Q1, Q2, Q3, Q4),$$

which proves the theorem.

In particular, if P, Q are the circular points at infinity, then the conic is a circle. And the points P1, Q2; P2, Q1 are the antifocal points of 1; 2; viz., calling these 1′, 2′, then 12 and 1′2′ are lines at right angles to each

other, having a common centre O, but such that $1'2' = i\,.\,12$, ($i = \sqrt{(-1)}$, as usual); or, what is the same thing, $O1 = O2 = i\,.\,O1' = i\,.\,O2'$. And the theorem is as follows: viz., if 1, 2, 3, 4 are points on a circle, and

1′, 2′ are the antifocal points of 1, 2,
3′, 4′ „ „ „ 3, 4,

then 1′, 2′, 3′, 4′ are points on a circle.

As an *à posteriori* proof, take the centre of the given circle as origin, so that (α_1, β_1), (α_2, β_2), (α_3, β_3), (α_4, β_4) being the coordinates of 1, 2, 3, 4, and the radius being taken as unity, we have

$$\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = \alpha_3^2 + \beta_3^2 = \alpha_4^2 + \beta_4^2 = 1.$$

Suppose for a moment that x, y are the coordinates of the antifocal points of 1, 2; we have

$$x - \alpha_1 \pm i(y - \beta_1) = 0, \quad x - \alpha_2 \mp i(y - \beta_2) = 0,$$
$$\text{that is} \quad x + iy = \alpha_1 + i\beta_1, \quad x - iy = \alpha_2 - i\beta_2,$$

for the coordinates of the one point; and similarly

$$x - iy = \alpha_1 - i\beta_1, \quad x + iy = \alpha_2 + i\beta_2,$$

for the coordinates of the other point.

Hence, taking the new coordinates

$$X = x + iy, \quad Y = x - iy,$$

and similarly $A_1 = \alpha_1 + i\beta_1$, $B_1 = \alpha_1 - i\beta_1$, &c.; the coordinates of the antifocal points 1′, 2′ are (A_1, B_2) and (A_2, B_1) respectively; but we have $A_1B_1 = \alpha_1^2 + \beta_1^2 = 1$, $A_2B_2 = \alpha_2^2 + \beta_2^2 = 1$; so that $B_1 = \dfrac{1}{A_1}$, $B_2 = \dfrac{1}{A_2}$; and the coordinates are $\left(A_1, \dfrac{1}{A_2}\right)$, $\left(A_2, \dfrac{1}{A_1}\right)$ respectively. Similarly the coordinates of the antifocal points (3′, 4′) are $\left(A_3, \dfrac{1}{A_4}\right)$, $\left(A_4, \dfrac{1}{A_3}\right)$ respectively.

Take as the equation of the circle through the two pairs of antifocal points

$$x^2 + y^2 + 2\lambda x + 2\mu y + \nu = 0,$$

or, what is the same thing,

$$XY + \lambda(X + Y) - i\mu(X - Y) + \nu = 0,$$
$$\text{that is,} \quad XY + LY + MX + N = 0,$$
$$\text{if} \quad L = \lambda + i\mu, \quad M = \lambda - i\mu, \quad N = \nu.$$

We ought then to have

$$\frac{A_1}{A_2} + L\frac{1}{A_2} + MA_1 + N = 0,$$
$$\frac{A_2}{A_1} + L\frac{1}{A_1} + MA_2 + N = 0,$$
$$\frac{A_3}{A_4} + L\frac{1}{A_4} + MA_3 + N = 0,$$
$$\frac{A_4}{A_3} + L\frac{1}{A_3} + MA_4 + N = 0;$$

and these will exist simultaneously, if

$$\begin{vmatrix} \dfrac{A_1}{A_2}, & \dfrac{1}{A_2}, & A_1, & 1 \\ \dfrac{A_2}{A_1}, & \dfrac{1}{A_1}, & A_2, & 1 \\ \dfrac{A_3}{A_4}, & \dfrac{1}{A_4}, & A_3, & 1 \\ \dfrac{A_4}{A_3}, & \dfrac{1}{A_3}, & A_4, & 1 \end{vmatrix} = 0,$$

an identical equation which is easily verified. It, in fact, gives

$$\left(\frac{1}{A_2}-\frac{1}{A_1}\right)\left(A_3-A_4\right)-\left(A_1-A_2\right)\left(\frac{1}{A_4}-\frac{1}{A_3}\right)+\left(\frac{A_1}{A_2}-\frac{A_2}{A_1}\right)\left(1-1\right)+$$

$$\left(1-1\right)\left(\frac{A_3}{A_4}-\frac{A_4}{A_3}\right)-\left(\frac{1}{A_2}-\frac{1}{A_1}\right)\left(A_3-A_4\right)+\left(A_1-A_2\right)\left(\frac{1}{A_4}-\frac{1}{A_3}\right)=0,$$

which is obviously true. The equation may also be written

$$\begin{vmatrix} 1, & A_1, & A_2, & A_1A_2 \\ 1, & A_2, & A_1, & A_1A_2 \\ 1, & A_3, & A_4, & A_3A_4 \\ 1, & A_4, & A_3, & A_3A_4 \end{vmatrix} = 0,$$

and in this form it expresses the known theorem of the equality of the anharmonic ratios of (A_1, A_2, A_3, A_4) and (A_2, A_1, A_4, A_3).

But, in order to actually find the circle, we may write

$$\begin{aligned} XY + LY + MX \quad &+ N \quad = 0, \\ A_1 + L \quad + MA_1A_2 &+ NA_2 = 0, \\ A_2 + L \quad + MA_1A_2 &+ NA_1 = 0, \\ A_3 + L \quad + MA_3A_4 &+ NA_4 = 0, \end{aligned}$$

and eliminating L, M, N, the equation of the circle is

$$\begin{vmatrix} XY, & Y, & X, & 1 \\ A_1, & 1, & A_1A_2, & A_2 \\ A_2, & 1, & A_1A_2, & A_1 \\ A_3, & 1, & A_3A_4, & A_4 \end{vmatrix} = 0,$$

or, reducing, this is

$$\left\{A_2-A_1\right\}\left\{XY(A_3A_4-A_1A_2)+Y\left[A_1A_2(A_3+A_4)-A_3A_4(A_1+A_2)\right]\right.$$
$$\left.+X(A_1+A_2-A_3-A_4)+(A_3A_4-A_1A_2)\right\}=0,$$

or say

$$XY(A_1A_2-A_3A_4)+Y\left\{A_3A_4(A_1+A_2)-A_1A_2(A_3+A_4)\right\}$$
$$+X\left\{(A_3+A_4)-(A_1+A_2)\right\}+(A_1A_2-A_3A_4)=0;$$

$$\text{that is,}\quad \begin{vmatrix} XY+1, & X, & Y \\ A_1+A_2, & A_1A_2, & 1 \\ A_3+A_4, & A_3A_4, & 1 \end{vmatrix} = 0,$$

which is the required equation; or, transforming to the original axes, we have $x+iy = X$, $x-iy = Y$, &c., and $\therefore XY = x^2+y^2$; and the equation becomes

$$\begin{vmatrix} x^2+y^2+1 & , x+iy & , x-iy \\ \alpha_1+\alpha_2+i(\beta_1+\beta_2), & (\alpha_1+i\beta_1)(\alpha_2+i\beta_2), & 1 \\ \alpha_3+\alpha_4+i(\beta_3+\beta_4), & (\alpha_3+i\beta_3)(\alpha_4+i\beta_4), & 1 \end{vmatrix} = 0,$$

which is the equation of the circle through the two pairs of antifocal points.

[NOTE.—The *second* form of the equation of the circle may be otherwise deduced from the *first*, without expanding the determinants, by the following method:—

$$\begin{vmatrix} XY, & Y, & X, & 1 \\ A_1, & 1, & A_1A_2, & A_2 \\ A_2, & 1, & A_1A_2, & A_1 \\ A_3, & 1, & A_3A_4, & A_4 \end{vmatrix} = \begin{vmatrix} XY+1, & Y, & X, & 1 \\ A_1+A_2, & 1, & A_1A_2, & A_2 \\ A_1+A_2, & 1, & A_1A_2, & A_1 \\ A_3+A_4, & 1, & A_3A_4, & A_4 \end{vmatrix} =$$

$$\begin{vmatrix} XY+1, & Y, & X, & 1 \\ A_1+A_2, & 1, & A_1A_2, & A_2 \\ 0, & 0, & 0, & A_1-A_2 \\ A_3+A_4, & 1, & A_3A_4, & A_4 \end{vmatrix} = (A_1-A_2)\begin{vmatrix} XY+1, & X, & Y \\ A_1+A_2, & A_1A_2, & 1 \\ A_3+A_4, & A_3A_4, & 1 \end{vmatrix};$$

$$\therefore \begin{vmatrix} XY+1, & X, & Y \\ A_1+A_2, & A_1A_2, & 1 \\ A_3+A_4, & A_3A_4, & 1 \end{vmatrix} = 0.$$

—EDITOR.]

II. *Solution by* MR. W. K. CLIFFORD.

1. Let the four points P1, Q2; P2, Q1; P3, Q4; P4, Q3 be called S, T, U, V, respectively; then

$$\{P.1234\} = \{Q.1234\};$$

$$\text{but } \{P.1234\} = \{P.STUV\}, \text{ and } \{Q.1234\} = \{Q.TSVU\},$$

$$\text{also } [TSVU] = [STUV], \therefore \{P.STUV\} = \{Q.STUV\},$$

which proves that the six points P, Q, S, T, U, V lie on a conic.

2. Let $A=0$, $B=0$, $C=0$, $D=0$, denote respectively the pairs of right lines (P1, Q1), (P2, Q2), (P3, Q3), (P4, Q4). Then we shall prove presently that there is an identical relation

$$A+B+C+D = 0,$$

constant multipliers being supposed.

3. The Jacobian of any three of the four conics A, B, C, D is obviously the original conic PQ1234, together with the straight line PQ. Now the conic $A+B=0$ is identical with $C+D=0$ (by Art. 1); and it passes through all the intersections of A with B, and of C with D. It must therefore be the very conic PQSTUV. And there are clearly two more conics, namely, $A+C$ or $D+B$, and $A+D$ or $B+C$, obtained just in the same way. It may be as well to remark that the three are represented by the equation

$$\frac{1}{A}+\frac{1}{B}+1+\frac{1}{D}=0.$$

Moreover, the original conic may be reproduced by treating STUV in the same way as we treated 1234. We have therefore four conics derived from the two points P, Q in a symmetrical manner. *Each of these conics is the Jacobian of the other three;* the line PQ being of course added. For the three conics $A+B$, $B+C$, $C+A$ have the same Jacobian as A, B, C, that is, the original conic and the line PQ.

The pole of PQ, with regard to the conic STUV, is the intersection of 12 and 34. For the chord ST is divided harmonically by PQ and 12, and the chord UV by PQ and 34. Hence *the poles of PQ, with regard to any three of the conics, form a self-conjugate triad with regard to the fourth.* For the poles with regard to $A+B$, $B+C$, $C+A$, are the intersections of (12, 34), (14, 23), (13, 42), which form a self-conjugate triad of any conic 1234.

4. By projecting the points P, Q into the circular points at infinity, we may prove M. Laguerre's theorems, proposed in the *Nouvelles Annales* for March 1864 (p. 141, Quest. 698) :—

"Lorsqu'une courbe a quatre foyers sur un cercle, elle en a nécessairement douze autres situés par quatre sur trois autres cercles; tous ces cercles sont orthogonaux entre eux."

It follows also that when four circles cut each other orthogonally, each centre is the intersection of perpendiculars of the triangle formed by the other three. Hence one of the circles must be imaginary.

5. We now proceed to prove the statement in (2).

LEMMA.—When four conics have the same Jacobian, three and three, their equations are connected by an identical linear relation.

Any four conics can be reduced simultaneously to the form

$$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0,$$

$$a_2x^2+b_2y^2+c_2z^2+d_2w^2=0,$$

$$\&c. \qquad \&c.$$

where $x+y+z+w=0$. This we can see by counting the constants. Let R be the determinant $(a_1b_2c_3d_4)$, and A_1, A_2, &c, its first minors. Then the Jacobians are

$$\frac{A_1}{x}-\frac{B_1}{y}+\frac{C_1}{z}-\frac{D_1}{w}=0,$$

$$\&c. \qquad \&c.$$

and if these are all identical, we must have $(A_1B_2C_3D_4)=0$, which implies that $(a_1b_2c_3d_4)=0$, or the conics are connected by an identical linear relation.

For a metric interpretation, see Dr. Salmon's *Conics*, 4th ed., Art. 94.

1454 (Proposed by MATTHEW COLLINS, B.A.)—Through a given point P, within a given angle BAC, to draw a straight line BPC so that the *Geometric*, *Harmonic*, or *Arithmetic* mean between the segments PB, PC may be given, or a minimum.

Solution by the PROPOSER.

1. When the *Geometric* mean between PB and PC is given, it is plain that PB . PC (= the square of the given Geometric mean) is given (Fig. 1). Now, when this rectangle and the point P, the *straight* line AB and the angle BPC are given, it is well known and very easily proved that the locus of the point C is a circle which will cut the line AC (whether this line be straight or curved) in a point C, which joined to P gives the required line BPC. In the particular case proposed, $\angle$ BPC = 2 right angles, and AC is *straight;* then there will obviously be two solutions, as a circle cuts a straight line in two points only; and then PB . PC will plainly be a minimum, when AB = AC, or when BPC is drawn through P parallel to the bisector of the supplement of A. For then a circle D can touch AB and AC at B and C, and if *any* other straight line $B'bPcC'$ passing through P cut this circle D at b and c, and cut the sides of the $\angle$ BAC at B′ and C′, we have (Euc. iii. 35)

$$PB.PC = Pb.Pc < PB'.PC'.$$

2. When the *Harmonic* mean between PB and PC is given, complete the parallelogram PB′AC′, and let B′DE, parallel to the required line BPC, cut the given straight line AP in D; then

$$BC : BP = B'E : B'D = PC : B'D;$$

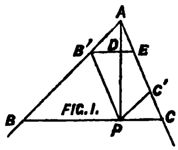


FIG. 1.

therefore $B'D \left(= \dfrac{PB.PC}{BC}\right)$ is half the *given* harmonic mean between PB and PC; and as the parallelogram AB′PC′ is given and fixed, we have only to inflect B′D, or half the given harmonic mean between the given point B′ and the given straight line AP, and then BPC drawn through P parallel to B′D will be the required line, and will manifestly (for the reason already assigned in Art. 1) admit of two positions. When the harmonic mean is a *minimum*, its half B′D will of course be a minimum, which will plainly happen when B′D, and therefore also BPC, which is parallel to B′D, is perpendicular to AP.

COROLLARY.—Hence, given of a triangle ABC its altitude (AP), vertical angle, and the harmonic mean between the segments (PB, PC) of the base made by the perpendicular, we can construct it: for then we have the base AP, the perpendicular B′D, and the vertical angle of the triangle APB′ given; and thence this triangle is easily constructed.

3. When BDC is required to touch a given circle (whose centre is P), and the arithmetic mean between DB and DC is given: Then it is plain that BC is given (say $= d$); draw CG and EPH perpendiculars to AB; put AE $= a$, CG $= x$, $\dfrac{CG}{AG} = \dfrac{e}{f}$ = tan A (which is given), PE $= b$, and PD $= c$.

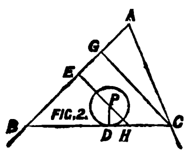


FIG. 2.

$$\text{Now } \frac{CG}{AG} = \frac{x}{AG} = \frac{e}{f}, \quad \therefore\ AB = AG + BG + \frac{fx}{e} + \sqrt{(d^2 - x^2)}.$$

Again the similar triangles PDH and BGC give

$$\frac{\text{PH}}{\text{PD}}=\frac{\text{BC}}{\text{BG}},\ \text{or}\ \text{PH}=\frac{\text{PD}\,.\,\text{BC}}{\text{BG}}=\frac{cd}{\sqrt{(d^2-x^2)}},$$

$$\therefore\ \text{EH}=\frac{cd}{\sqrt{(d^2-x^2)}}+b,$$

and this, multiplied by $\frac{\text{BE}}{\text{EH}}=\frac{\text{BG}}{\text{CG}}=\frac{\sqrt{(d^2-x^2)}}{x}$, gives

$$\text{BE}=\frac{cd+b\sqrt{(d^2-x^2)}}{x},$$

$$\therefore\ \text{AB}=\text{AE}+\text{EB}=a+\frac{cd+b\sqrt{(d^2-x^2)}}{x}=\frac{fx}{e}+\sqrt{(d^2-x^2)},$$

which, cleared of surds and reduced, gives

$$(e^2+f^2)\,x^4-2ex^3\,(af+be)+ex^2\,(a^2e+b^2e-d^2e-2cdf)+2de^2x\,(ac+bd)$$
$$+\,d^2e^2\,(c^2-b^2)=0.$$

(See Prob. 33, Newton's *Universal Arithmetic.*)

This equation is not much simplified when $c=0$, *i. e.*, when BC passes through P; but this position of BPC is then easily obtained by means of a *conchoid* whose pole is P and axis is AB, whether AC be straight or curved (see Fig. 90, Newton's *Universal Arithmetic*) just as, in case 1, the position of BPC was obtained by means of a *circle*, whether the other given line AC be straight or curved.

When AC is straight as well as AB, then BPC will be a minimum, when the point B is as far from P as C is from the foot of the perpendicular from A upon BPC, as is demonstrated in the Solution of Quest. 632, where may also be seen a method of finding the position of BPC (a minimum) by means of a hyperbola.

Lastly, when the given point P lies in the middle of the given angle BAC (or in the middle of its supplement), the question is *then* easily solved *geometrically*; since in the triangle ABC we have given the base BC, the vertical angle A, and the bisector AP of the internal (or external) vertical angle, from which data this triangle is constructed as follows:—

Let AP cut in D the circle about the triangle ABC; then $\angle$ DBC = DCB = $\frac{1}{2}$BAC; therefore all the angles and one side BC of the triangle DBC are given, hence the other sides DB = DC are given; but DA . DP = DB^2, and thus AP is to be cut externally at D so that the rectangle (DA . DP) of its segments may be given; then inflect the given line DB between the point D (already found) and the given straight line AB.

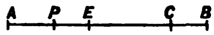
[NOTE.—The locus in Art. 1 is equivalent to the following:—

The area and an angle of a triangle are given in *magnitude*, and the given angle turns round its vertex which is at a fixed point, while another vertex of the triangle moves along a fixed straight line; then the *locus* of the third vertex will be a *circle* passing through the fixed point.

For suppose P to be the fixed point, AB the fixed straight line, BPC *any* position of the triangle, and B′PC′ *that* position in which B′P is perpendicular to AB (the figure may be easily drawn); then $\angle$ BPC = B′PC′ and Δ BPC = B′PC′ (or, what amounts to the same thing, BP . PC = B′P . PC′ = a *constant*), therefore BP : B′P = PC′ : PC (Euc. vi. 15), also $\angle$ BPB′ = CPC′, therefore the triangle CPC′ is similar to BPB′ (Euc. vi. 6); hence PCC′ is a *right angle*, and the locus of C is the circle on PC′ as diameter.—EDITOR.]

1465 (Proposed by W. O. PHILLIPS, M.A.)—C is a given point in the given line AB; find the position of a point P in AC, such that $PC^2 : PB . AB$ may be equal to a given ratio.

Solution by R. TUCKER, M.A.; Mr. J. CONWILL; Mr. J. WILSON; Mr. T. POOLEY; *and the* PROPOSER.



Let $PC^2 = n . PB . AB$, where n is the given ratio: in CA take $CE = n . AB$; then

$$PC^2 = PB . CE = PC . CE + BC . CE \text{ (Euc. ii. 1)};$$

$$\text{hence } PC . PE = BC . CE = \text{a known area},$$

and P is readily found. (See McDowell's *Exercises*, Prop. 60.)

1476 (Proposed by Mr. J. WILSON.)—Find the equation of the curve subject to the condition $ps = qr$, where p, q, r, s are the first, second, third, and fourth differential coefficients of the ordinate.

Solution by MATTHEW COLLINS, B.A.; R. TUCKER, M.A.; *and the* PROPOSER.

$$\text{Since } \frac{q}{p} = \frac{s}{r}, \quad \therefore \frac{q\,dx}{p} = \frac{s\,dx}{r} \text{ or } \frac{dp}{p} = \frac{dr}{r},$$

$$\text{whose integral is } \log p + \log C = \log r, \text{ or } r = Cp;$$

$$\text{hence } r\,dx = Cp\,dx \text{ or } dq = C\,dy,$$

$$\text{whose integral is } q = Cy + D;$$

and thence, by muliplying by $p\,dx = dy$, we get $pq\,dx = p\,dp = Cy\,dy + D\,dy$,

$$\text{whose integral is } p^2 = \frac{dy^2}{dx^2} = Cy^2 + 2Dy + E,$$

$$\therefore \pm dx = \frac{dy}{(Cy^2 + 2Dy + E)^{\frac12}}, \text{ or } \pm C^{\frac12}\,dx = \frac{C\,dy}{\left\{(Cy + D)^2 + CE - D^2\right\}^{\frac12}},$$

$$\text{whose integral is } C^{\frac12}x = \log\left\{Cy + D + (C^2y^2 + 2CDy + CE)^{\frac12}\right\},$$

$$\text{or } e^{C^{\frac12}x} = Cy + D + (C^2y^2 + 2CDy + CE)^{\frac12}$$

$$\text{whose reciprocal gives } e^{-C^{\frac12}x} = \frac{Cy + D - (C^2y^2 + 2CDy + CE)^{\frac12}}{D^2 - CE},$$

$$\therefore\ 2(Cy+D)=e^{C^{\frac{1}{2}}x}+(D^2-CE)\,e^{-C^{\frac{1}{2}}x},$$

which is plainly of the form $y+c=Ae^{mx}+Be^{-mx}$,

the equation of the required curve, which includes the *Catenary* as a particular curve of the series. When C is *negative*, and therefore $m\ (=C^{\frac{1}{2}})$ *imaginary*, the equation becomes

$$y+c=A\cos mx+B\sin mx,$$

another form of the equation of the required curve;—and these two equations have plainly all the necessary generality, as each of them has *four* arbitrary constants A, B, c, m.

COROLLARY.—If p, q, r were required to be in *Geometrical Progression*, by proceeding in a similar way we should easily find $ay+b=c^x$ as the equation of the required curve, in its most general form.

1499 (Proposed by J. M. WILSON, M.A., Rugby.)—Two men A and B sat down under a tree to dine together, A contributing three loaves and a cold fowl, B two loaves and half a bottle of wine. A traveller C coming up, they invite him to share with them, and having fared alike, and finished all their provisions, the traveller, pleased with their hospitality, offered them all the money he had about him, viz. 6s. 7d., to be divided equitably. A, after consideration, proposed that B should have 2s. $1\frac{1}{4}$d., and he the rest; but B observed that his half bottle of wine was worth more than A's fowl by three farthings, and showed that, if he paid for what he had eaten of A's, and was paid for what he had given away at the same rate, he would have received 2s. 2d.

Find the prices of a loaf, a fowl, and a bottle of wine.

N.B.—This problem was suggested by one in Bland's *Algebraical Problems* about two Spanish muleteers, the solution of which is erroneous in principle.

I. *Solution by the* PROPOSER.

Let x, y, $y+3$ denote the respective prices of a loaf, a fowl, and half a bottle of wine, all expressed in farthings. Then the traveller pays in proportion to what they gave him,

$$\therefore\ \text{A's debt : B's debt} = \frac{4x}{3}+\frac{y}{3} : \frac{x}{3}+\frac{y+3}{3},$$

$$\therefore\ \frac{x+y+3}{5x+2y+3}=\frac{\text{2s. 1d.}}{\text{6s. 7d.}}=\frac{25}{79}\ \dots\dots\dots\dots\dots\dots\ (1),$$

taking off the farthing which A owes B.

On B's theory

$$\text{A's debt : B's debt} = \frac{4x}{3} + \frac{2y}{3} - \frac{y+3}{3} : \frac{x}{3} + \frac{2y+6}{3} - \frac{y}{3},$$

$$\therefore \frac{x+y+6}{5x+2y+3} = \frac{26}{79} \quad \text{.......... .... (2).}$$

From (1) and (2) we have $\dfrac{3}{5x+2y+3} = \dfrac{1}{79}$,

$$\therefore x+y+3 = 75, \text{ and } 5x+2y+3 = 237;$$

whence $x=30$, $y=42$; and therefore the price of a loaf = 7½d., of a fowl = 10½d., and of a bottle of wine = 1s. 10½d.

II. *Solution by* Mr. S. BILLS.

The two theories of equitable division may be expressed, perhaps a little more clearly, as follows:—"The half bottle of wine being worth three farthings more than the fowl, A, considering himself indebted to B one farthing for the difference in value of the portions of wine and fowl supplied by the one to the other, proposed that B should have 2s. 1¼d., and he the rest. But B objected to this, and showed that if he paid A for what he had eaten of A's, and was paid for what he had supplied to A and C at the *same* rate, he ought to receive 2s. 2d."

Let, then, x denote the price of a loaf, y that of the fowl, and z that of the half bottle of wine, all in farthings. Also, let r be the rate at which B is to pay A for what he receives of him and for what he supplies to A and C; that is, suppose B to give for what he receives from A, and to receive for what he supplies to A and C, r times its *intrinsic value*.

According to A's proposal, B is entitled to 2s. 1d. of the 6s. 7d., independently of the farthing which A owes B; we shall thus be led to the relation

$$\frac{4x}{3} + \frac{y}{3} : \frac{x}{3} + \frac{z}{3} = 54 : 25 \quad \text{.................... (1).}$$

Next, on B's theory, he is to pay $\frac{1}{3}ry$ for what he receives from A; and he will have to receive $\frac{1}{3}r(x+2z)$ for what he supplies to A and C. The difference of these, according to B's showing, should be equal to 2s. 2d., or 104 farthings;

$$\therefore r\left(\frac{x}{3} + \frac{2z}{3} - \frac{y}{3}\right) = 104 \quad \text{.................... (2).}$$

But we have evidently, by the question,

$$r\left(\frac{5x}{3} + \frac{y}{3} + \frac{z}{3}\right) = 316 \text{.......... (3); and } z = y+3 \text{.......... (4).}$$

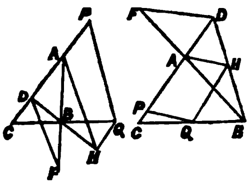
From (1), (2), (3), (4) we readily find $x=30$, $y=42$, $z=45$, and $r=4$.

The respective prices of a loaf, a fowl, and a bottle of wine are, therefore, 7½d., 10½d., and 1s. 10½d.; and C pays for what he eats *four times* its *intrinsic* value.

1477 (Proposed by R. Tucker, M.A.)—A and B are given points on the legs of a given angle ACB; it is required to draw a transversal PQ, cutting AC, CB in P, Q, so that AP : PQ : QB = $m : n : p$; $m : n : p$ being given ratios.

Solution by Mr. W. Hopps; Rev. R. Townsend, M.A.; Mr. T. Pooley; Matthew Collins, B.A.; Mr. J. Wilson; Mr. A. Renshaw; *and the* Proposer.

On CA take CD, making CD : CB = $m : p$; and from D inflect DF to meet the line through AB at F, making CD : DF = $m : n$. Draw AH parallel to DF to meet the line through D, B at H; also draw HQ, QP respectively parallel to CD, AH. PQ is the transversal required.



For, by parallels, similar triangles, and construction,

$$AP : CD = HQ : CD = HB : DB = HA : DF = PQ : DF;$$

$$\therefore\ AP : PQ = CD : DF = m : n.$$

$$\text{Also,}\ AP : QB = HQ : QB = CD : CB = m : p.$$

$$\text{Hence}\ AP : PQ : QB = m : n : p.$$

1494 (Proposed by G. T. Sadler, F.R.A.S.)—Find x and y from the equations

$$x^2y + xy^2 = 7 \dots\dots (1),$$

$$x^7y^2 + x^2y^7 = 1608\tfrac{37}{64} \dots\dots (2).$$

Also find x, y, z from the equations

$$(x^{15}+y^{15}+z^{15})^3 + (x^3+y^3)^2 = 31 \dots\dots (1),$$

$$(x^{15}+y^{15}+z^{15})^3 (x^3+y^3+z^3)^3 = 729 \dots\dots (2),$$

$$(x^3+y^3)^2 + (x^3+y^3+z^3)^3 = 31 \dots\dots (3).$$

Solution by Mr. S. Bills; R. J. Nelson, M.A.; W. Easterby, B.A.; Mr. J. Wilson; Mr. A. Renshaw; *and the* Proposer.

Taking the first part of the question, put $x+y=s$, and $xy=p$; then we have

$$ps = 7 \dots\dots (3), \quad \text{and} \quad p^2s^5 - 5p^3s^3 + 5p^4s = 1608\tfrac{37}{64} \dots\dots (4).$$

From (3), $p = \dfrac{7}{s}$; and by substituting this value of p in (4) and reducing,

we obtain $s^6 - 67\frac{53}{64}s^3 = -245$; whence $s^3 = 64$ or $\frac{245}{64}$. Taking the former value, we have $s=4$, and thence $p=\frac{7}{4}$. We have now $x+y=4$ and $xy=\frac{7}{4}$; whence we obtain $x=\frac{7}{2}$ and $y=\frac{1}{2}$. Another answer may be obtained from the value $s^3=\frac{245}{64}$.

Again, for the second part of the question, put $x^3=u$, $y^3=v$, $z^3=w$; then

$$(u^5+v^5+w^5)^3+(u+v)^2 = 31 \text{.........} (1),$$

$$(u^5+v^5+w^5)^3(u+v+w)^3 = 729 \text{.........} (2),$$

$$(u+v)^2+(u+v+w)^3 = 31 \text{.........} (3).$$

Subtracting (3) from (1), we have $(u^5+v^5+w^5)^3 = (u+v+w)^3$........(4).

Substituting this result in (2) and extracting the root, we find

$u+v+w=3$......(5). From (3) and (5) we find $u+v=2$......(6).

(5)−(6) gives $w=1$. Substituting this value of w in (4), and extracting the cube root, we have $u^5+v^5=u+v$, or

$$u^4-u^3v+u^2v^2-uv^3+v^4 = 1 \text{.........} (7).$$

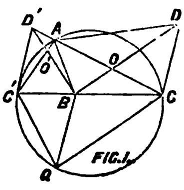
By means of (6) we may easily reduce (7) to the form $u^2v^2-4uv=-3$, whence we find $uv=1$, or 3. Taking the value $uv=1$, and combining it with (6), we find $u=v=1$; from which we find $x=y=z=1$, which numbers satisfy the given equations.

1490 (Proposed by the EDITOR.)—Construct a quadrilateral, having given two opposite sides, the angles they make with the third side, and the angle at which the diagonals intersect.

In particular, examine the case in which the three given angles are together equal to two right angles.

Also show that, if the third side is perpendicular to the two given sides, the rectangle contained by these two sides is equal to that contained by the third side and the segment intercepted on it by two perpendiculars to the diagonals drawn through their point of intersection.

Solution by MR. A. RENSHAW; *and the* PROPOSER.



1. At the point B (Fig. 1) in the straight line BC, make the angles CBA, CBQ, on opposite sides of BC, equal to the two given angles of the quadrilateral; also take BA, BQ equal to the two given sides. Through A, Q (Euc. iii. 33) describe a circular segment containing an angle equal to the given one contained by the diagonals, and cutting BC in C and C′. Join QC and QC′. Through B draw BD, BD′ parallel respectively to QC, QC′, and meeting CD, C′D′, parallels through

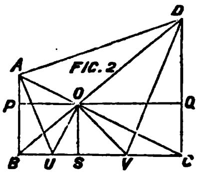
C and C′ to BQ, in D and D′. Join AD, AD′; then each of the quadrilaterals ABCD, ABC′D′ satisfies the conditions of the problem.

The demonstration is evident from the construction.

The points C, C′ will be on the *same* or on *opposite* sides of B, according as the three *given* angles are together *less* or *greater* than two right angles; and the quadrilateral will be *convex*, *reentrant*, or *intersecting*, according to the relations amongst the data.

2. When the three given angles (ABC, BCD or CBQ, AOB or ACQ) are together equal to two right angles, the point C′ will *coincide* with B (Euc. iii. 22), and there will be only *one* quadrilateral satisfying the conditions of the problem.

3. When ABC and BCD are right angles (Fig. 2); through O, the point of intersection of the diagonals, draw PQ, OS, OU, OV respectively perpendicular to AB, BC, AC, BD; then, by similar triangles, we have



$$AP : OS = OS : DQ, \quad \therefore AP \cdot DQ = OS^2;$$
$$AP : OP = US : OS, \quad \therefore AP \cdot QC = US \cdot OP;$$
$$DQ : OQ = VS \cdot OS, \quad \therefore PB \cdot DQ = VS \cdot OQ;$$

$$\text{also } PB^2 = OS^2 = US \cdot CS = VS \cdot BS;$$
$$\therefore AP \cdot CD + PB \cdot CD = US \cdot BC + VS \cdot BC,$$
$$\text{that is, } AB \cdot CD = BC \cdot UV.$$

4. To obtain a solution by *Trigonometry*, let α, β, γ denote the *given* angles ABC, BCD, AOB (Fig. 1), and θ the *unknown* angle BAC; then putting $\alpha+\beta+\gamma=\sigma$, $\alpha+\gamma=\tau$, $\alpha+\sigma=\upsilon$, we have

$$\frac{a \sin\theta}{\sin(\theta+\alpha)} = BC = \frac{b \sin(\theta+\sigma)}{\sin(\theta+\tau)}, \text{ or}$$

$(a\cos\tau - b\cos\alpha\cos\sigma)\tan^2\theta + (a\sin\tau - b\sin\upsilon)\tan\theta = b\sin\alpha\sin\sigma \ldots\ldots (i)$, whence θ can be found, and thus the quadrilateral is determined.

5. Otherwise, putting $BC = x$, we have

$$\tan^{-1}\left(\frac{a\sin\alpha}{x-a\cos\alpha}\right) + \tan^{-1}\left(\frac{b\sin\beta}{x-b\cos\beta}\right) = \gamma,$$

$$\text{whence } x^2 - cx + ab\sin\sigma\,\mathrm{cosec}\,\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots (ii),$$
$$\text{where } c = \left\{a\sin(\alpha+\gamma) + b\sin(\beta+\gamma)\right\}\mathrm{cosec}\,\gamma;$$

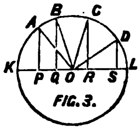
$$\therefore x = BC = \tfrac{1}{2}c\left\{1 \pm \sqrt{(1-4\,abc^{-2}\sin\sigma\,\mathrm{cosec}\,\gamma)}\right\} \ldots\ldots\ldots (iii),$$

which determines the quadrilateral.

6. From (*i*), (*ii*), or (*iii*) we see that there are in general *two* quadrilaterals satisfying the conditions of the problem; if however $\sigma=\pi$ or 2π, so that $\sin\sigma=0$, there is but *one*: which agrees with Art. 2.

7. A simple representation may be given of c in Art. 5. Thus suppose Fig. 2 to represent a quadrilateral *of any species*, and let circles drawn round AOB, DOC meet BC in U, V respectively; then clearly $\angle AUB = DVC = \gamma$ and $c = BU + CV$. When, therefore, ABC and BCD are together *equal to*

two right angles, we have $\sin\sigma = -\sin\gamma$, and (*ii*) becomes $x(x-c) = ab$, whence we obtain the relation AB . CD = BC . UV, which includes that in Art. 3 as a particular case, since ABU, AOU, DCV, DOV are then *right angles*.



8. From Arts. 1 and 4 we may obtain solutions of the following problem. Let A, B, C, D (Fig. 3) be fixed points on a circle, to draw a diameter KOL such that, perpendiculars AP, BQ, CR, DS being drawn thereon, DS . BQ : CR . AP may be equal to a *given* ratio $a : b$. For, putting $\angle AOB = \alpha$, $BOC = \gamma$, $COD = \beta$, $AOK = \theta$, the condition is expressed by the equation in Art. 4, which may be constructed as in Art. 1.

1505 (Proposed by Professor CAYLEY.)—If P, Q, 1, 2, 3, 4 be points on a conic; then the four points P1, Q2; P2, Q1; P3, Q4; P4, Q3 lie on a conic passing through the points P and Q.

Solution by WILLIAM SPOTTISWOODE, M.A., F.R.S.

If the points P, Q, 4, be taken for the vertices of the triangle of reference, the equation of the conic passing through P, Q, 1, 2, 3, 4, will be

$$\begin{vmatrix} y_1z_1, & z_1x_1, & x_1y_1 \\ y_2z_2, & z_2x_2, & x_2y_2 \\ y_3z_3, & z_3x_3, & x_3y_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equations of the lines (P1), (Q2), (P2), (Q1), (P3), (Q4), (P4), (Q3), are

$$\text{P1}\ldots\begin{vmatrix} \eta, & \zeta \\ y_1, & z_1 \end{vmatrix} = 0; \quad \text{P2}\ldots\begin{vmatrix} \eta, & \zeta \\ y_2, & z_2 \end{vmatrix} = 0; \quad \text{P3}\ldots\begin{vmatrix} \eta, & \zeta \\ y_3, & z_3 \end{vmatrix} = 0; \quad \text{P4}\ldots\eta = 0;$$

$$\text{Q2}\ldots\begin{vmatrix} \xi, & \zeta \\ x_4, & z_2 \end{vmatrix} = 0; \quad \text{Q1}\ldots\begin{vmatrix} \xi, & \zeta \\ x_1, & z_1 \end{vmatrix} = 0; \quad \text{Q4}\ldots\xi = 0; \quad \text{Q3}\ldots\begin{vmatrix} \xi, & \zeta \\ x_3, & z_3 \end{vmatrix} = 0;$$

and the coordinates of the points of intersection of these lines two and two are

$$\text{(P1) and (Q2)}\ \frac{\xi_1}{x_2z_1} = \frac{\eta_1}{y_1z_2} = \frac{\zeta_1}{z_1z_2}; \quad \text{(P2) and (Q1)}\ \frac{\xi_2}{x_1x_2} = \frac{\eta_2}{y_2z_1} = \frac{\zeta_2}{z_1z_2};$$

$$\text{(P3) and (Q4)}\ \xi_3 = 0,\ \frac{\eta_3}{y_3} = \frac{\zeta_3}{z_3}; \qquad \text{(P4) and (Q3)}\ \eta_4 = 0,\ \frac{\xi_4}{x_3} = \frac{\zeta_4}{z_3};$$

whence, the equation of the conic passing through the above 4 points and P (coordinates x_p, 0, 0) and Q (coordinates 0, y_q, 0) is

$$\begin{vmatrix} x_2^2 z_1^2, & y_1^2 z_1^2, & z_1^2 z_2^2, & y_1 z_1 z_2^2, & x_2 z_1^2 z_2, & x_2 y_1 z_1 z_2 \\ x_1^2 z_2^2, & y_2^2 z_1^2, & z_1^2 z_2^2, & y_2 z_1^2 z_2, & x_1 z_1 z_2^2, & x_1 y_2 z_1 z_2 \\ 0, & y_3^2, & z_3^2, & y_3 z_3, & 0, & 0 \\ x_3^2 & 0, & z_3^2, & 0, & x_3 z_3, & 0 \\ x_p^2, & 0, & 0, & 0, & 0, & 0 \\ 0, & y_q^2 & 0, & 0, & 0, & 0 \end{vmatrix} = 0$$

$$= x_p^2 y_q^2 z_1^2 z_2^2 z_3^2 \begin{vmatrix} z_1 z_2, & y_1 z_2, & x_2 z_1, & x_2 y_1 \\ z_1 z_2, & y_2 z_1, & x_1 z_2, & x_1 y_2 \\ z_3, & y_3, & 0, & 0 \\ z_3, & 0, & x_3 & 0 \end{vmatrix}$$

which, on reduction, is equivalent to (1). Hence the theorem in question.

1509 (Proposed by Dr. BOOTH, F.R.S.)—If $V \equiv F(x, y, z) = 0$ and $U \equiv \Phi(\xi, v, \zeta) = 0$ be the projective and tangential equations of the same surface, show that we may pass from the one to the other by the help of the following equations :—

$$\xi = \frac{dV}{dx} \div \left(\frac{dV}{dx} x + \frac{dV}{dy} y + \frac{dV}{dz} z\right), \text{ with similar expressions for } v \text{ and } \zeta;$$

$$x = \frac{dU}{d\xi} \div \left(\frac{dU}{d\xi} \xi + \frac{dU}{dv} v + \frac{dU}{d\zeta} \zeta\right), \text{ with similar expressions for } y \text{ and } z.$$

Apply these formulæ to find the tangential equations of the Lemniscate and Cissoid.

Solution by the PROPOSER.

1. If $V \equiv F(x, y, z) = 0$ be the projective equation of a surface, the equation of the tangent plane to it passing through the point (x', y', z') is

$$\frac{dV}{dx'}(x' - x) + \frac{dV}{dy'}(y' - y) + \frac{dV}{dz'}(z' - z) = 0.$$

Putting $z = 0$, $y = 0$, x will be $\frac{1}{\xi}$; hence we have

$$\xi = \frac{dV}{dx} \div \left(\frac{dV}{dx} x + \frac{dV}{dy} y + \frac{dV}{dz} z\right); \text{ and similar expressions for } v \text{ and } \zeta.$$

By eliminating x, y, z from these three equations and the *projective* equation (V) we should obtain the *tangential* equation of the surface.

2. Let $\mathrm{U} \equiv \Phi\,(\xi, \upsilon, \zeta) = 0$ be the tangential equation of a surface; then, the general relation between projective and tangential coordinates being $x\xi + y\upsilon + z\zeta = 1$, a slight change in ξ, υ, ζ will not affect the projective coordinates x, y, z of the point of contact of the tangent plane, hence

$$x d\xi + y d\upsilon + z d\zeta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating the equation $\mathrm{U}=0$, we have

$$\frac{d\mathrm{U}}{d\xi}\,d\xi + \frac{d\mathrm{U}}{d\upsilon}\,d\upsilon + \frac{d\mathrm{U}}{d\zeta}\,d\zeta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Comparing (2) with (1) multiplied by a constanct c, we have

$$cx = \frac{d\mathrm{U}}{d\xi},\quad cy = \frac{d\mathrm{U}}{d\upsilon},\quad cz = \frac{d\mathrm{U}}{d\zeta};$$

$$\text{hence } c\,(x\xi + y\upsilon + z\zeta) = c = \frac{d\mathrm{U}}{d\xi}\,\xi + \frac{d\mathrm{U}}{d\upsilon}\,\upsilon + \frac{d\mathrm{U}}{d\zeta}\,\zeta;$$

$\therefore x = \dfrac{d\mathrm{U}}{d\xi} + \left(\dfrac{d\mathrm{U}}{d\xi}\,\xi + \dfrac{d\mathrm{U}}{d\upsilon}\,\upsilon + \dfrac{d\mathrm{U}}{d\zeta}\,\zeta\right)$; and similar expressions for υ and ζ.

By eliminating ξ, υ, ζ from these three equations and the *tangential* equation (U) we may obtain the *projective* equation of the surface.

3. Let the equation of the *Lemniscate* in *projective* coordinates be $\mathrm{V} \equiv (x^2+y^2)^2 - 4a^2\,(x^2-y^2) = 0$; then putting $x^2+y^2 = r^2$ we have

$$4a^2\,(x^2-y^2) = r^4,\quad 8a^2x^2 = r^2\,(4a^2+r^2),\quad 8a^2y^2 = r^2\,(4a^2-r^2);$$

$$\therefore\ \xi = \frac{d\mathrm{V}}{dx} \div \left(\frac{d\mathrm{V}}{dx}\,x + \frac{d\mathrm{V}}{dy}\,y\right) = \frac{2x}{r^4}\,(r^2-2a^2),\quad \upsilon = \frac{2y}{r^4}\,(r^2+2a^2).$$

Substituting in these expressions the values of x and y in terms of r, we get

$$2a^2r^6\xi^2 = (4a^2+r^2)\,(r^2-2a^2)^2 = 16a^6 - 12a^4r^2 + r^6,$$
$$2a^2r^6\upsilon^2 = (4a^2-r^2)\,(r^2+2a^2)^2 = 16a^6 + 12a^4r^2 - r^6;$$
$$\therefore\ r^6\,(\xi^2+\upsilon^2) = 16a^4,\quad a^2r^4\,(\xi^2-\upsilon^2) = r^4 - 12a^4;$$

and eliminating r from these two equations, we get finally

$$27a^4\,(\xi^2+\upsilon^2)^2 = 4\,[1 - a^2\,(\xi^2-\upsilon^2)]^3,$$

the *tangential* equation of the *Lemniscate*.

4. Let the *projective* equation of the *Cissoid* be $\mathrm{V} \equiv x\,(x^2+y^2) - ay^2 = 0$; then as in Art. 3 we find $\xi = \dfrac{3a-2x}{ax}$, $\upsilon^2 = \dfrac{4\,(a-x)^3}{a^2x^3}$; hence, eliminating x from these two equations, we get

$$27a^2\upsilon^2 = 4\,(a\xi - 1)^3,$$

the *tangential* equation of the *Cissoid*.

[NOTE.—As an example of the method of finding the *projective* equation from the *tangential* equation, let us take the catacaustic of the circle, for parallel rays; then, by putting $\gamma = 0$ in equation (4) of the Solution of Question 1492, the *tangential* equation is found to be

$$\mathrm{U} \equiv 4a^2\,(1 - a^2\xi^2)\,(\xi^2+\upsilon^2) - 1 = 0.$$

Now assume $\sin^2\psi = 2a^2(\xi^2+v^2)$, then from $U=0$ we have

$$2a^2\xi^2 = 1 - \cot^2\psi,\quad 2a^2v^2 = \cos^2\psi\cot^2\psi;$$

$$\text{also}\ \frac{dU}{d\xi} = 8a^2\xi\left(1-2a^2\xi^2-a^2v^2\right) = 4\,a^2\xi\left(\cos^2\psi+\cot^2\psi\right),$$

$$\text{and}\ \frac{dU}{dv} = 8a^2v\left(1-a^2\xi^2\right) \qquad = 4\,a^2v\,\mathrm{cosec}^2\psi;$$

$$\therefore\quad \frac{dU}{d\xi}\,\xi + \frac{dU}{dv}\,v = 4\cos^2\psi;$$

hence, applying the formulæ in Art. 2, we have

$$2\left(x^2-a^2\right) = 3a^2\,\mathrm{cosec}^2\psi - \mathrm{cosec}^6\psi,\quad 2y^2 = a^2\,\mathrm{cosec}^6\psi;$$

and, eliminating ψ, the *projective* equation is

$$4\left(x^2+y^2-a^2\right)^3 = 27a^4y^2,$$

which shows that the *catacaustic of the circle, for parallel rays, is a two-cusped epicycloid,* whose base is concentric with the reflecting circle and has its radius (a) half the radius of that circle.

We may further observe that the tangential equation of the *Cardioid,* or *one-cusped epicycloid,* referred to the centre of the base as origin, and the radius (a) through the cusp as positive axis of ξ, is

$$27a^2(1-a\xi)\left(\xi^2+v^2\right) = 4.$$

This may be obtained from the equation in Question 1492 by putting $-\gamma=\rho=(3a)^{-1}$; for it can be easily shown (see Parkinson's *Optics,* Art. 72) that when the radiant point is in the *circumference* of the reflecting circle $(\gamma=-\rho)$ the caustic is a *Cardioid,* the radius (a) of whose base is one-third of that of the reflecting circle.—EDITOR.]

1513 (Proposed by the Rev. J. BLISSARD, B.A.)—Prove the following formulæ :—

$$(1.)\ \ldots\ldots\ldots\ldots\ \frac{(x-1)(x-2)\ldots(x-n)}{x(x+1)\ldots(x+n-1)} =$$

$$1+(-)^n\left\{n\,.\,\frac{1}{x}-\frac{n\left(n^2-1^2\right)}{1^2}\,.\,\frac{1}{x+1}+\frac{n\left(n^2-1^2\right)\left(n^2-2^2\right)}{1^2\,.\,2^2}\,.\,\frac{1}{x+2}-\&c.\right\}$$

(2.) The above formula expressed as $\dfrac{(\Gamma x)^2}{\Gamma(x-n)\,\Gamma(x+n)} =$

$$1-\frac{n^2}{1}\,.\,\frac{1}{x}+\frac{n^2\left(n^2-1^2\right)}{1\,.\,2}\,.\,\frac{1}{x(x+1)}-\frac{n^2\left(n^2-1^2\right)\left(n^2-2^2\right)}{1\,.\,2\,.\,3}\,.\,\frac{1}{x(x+1)(x+2)}+\&c.$$

and show that this equation is subject to the sole restriction that when n is not integral x must not be negative.

Solution by Professor Cayley; *and* X. U. J.

Let n be a positive integer, and suppose that $[x]^n$ denotes as usual the factorial $x(x-1)\ldots.(x-n+1)$; then we have

$$[x+k]^n = (1+\Delta)^k [x]^n = \left(1+k\Delta+\frac{k(k-1)}{1.2}\Delta^2+\&c.\right)[x]^n$$
$$= [x]^n + \frac{kn}{1}[x]^{n-1} + \frac{k(k-1)n(n-1)}{1.2}[x]^{n-2} + \&c.$$

Or putting $k=-n$ we have

$$[x-n]^n = [x]^n - \frac{n^2}{1}[x]^{n-1} + \frac{n^2(n^2-1^2)}{1.2}[x]^{n-2} - \&c.$$

And writing herein $(x+n-1)$ for x, and dividing by $[x+n-1]^n$, we have

$$\frac{[x-1]^n}{[x+n-1]^n} = 1 - \frac{n^2}{1}\cdot\frac{1}{x} + \frac{n^2(n^2-1^2)}{1.2}\cdot\frac{1}{x(x+1)} - \&c.;$$

or what is the same thing

$$\frac{(\Gamma x)^2}{\Gamma(x-n)\,\Gamma(x+n)} = 1 - \frac{n^2}{1}\cdot\frac{1}{x} + \frac{n^2(n^2-1^2)}{1.2}\cdot\frac{1}{x(x+1)} - \&c.,$$

which is the formula (2). The foregoing demonstration applies to the case of n a positive integer; but as the two sides are respectively unaltered when n is changed into $-n$, it is clear that the formula holds good also for n a negative integer. The right hand side is the hypergeometric series $F(n, -n, x, 1)$ and the formula therefore is

$$\frac{(\Gamma x)^2}{\Gamma(x-n)\,\Gamma(x+n)} = F(n, -n, x, 1),$$

a particular case of the known formula

$$\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} = F(\alpha, \beta, \gamma, 1),$$

which when α or β is a positive integer is a mere identity, true therefore for all values of γ; but if neither α nor β is a positive integer, then the right hand side is an infinite series which is only convergent for $\gamma > \alpha+\beta$. In the particular case we have $\alpha=n$, $\beta=-n$, $\gamma=x$; hence if n be a positive or negative integer, the formula is an identity, but if n be fractional, the condition of convergency is $x>0$, that is, x must be positive.

To prove the formula (1) it is only necessary to remark, that (n being a positive integer) the quantity $\dfrac{[x-1]^n}{[x+n-1]^n}$ is a rational fraction, the numerator and denominator whereof are of the same degree n, and which becomes $=1$ for $x=\infty$. Hence, decomposing it into simple fractions, we may write

$$\frac{[x-1]^n}{[x+n-1]^n} = 1 + S_r\,.\,\frac{A_r}{x+r}$$

where the summation extends from $r=0$ to $r=n-1$ both inclusive. And we have

$$A_r = \left\{\frac{(x+r)\,[x-1]^n}{[x+n-1]^n}\right\}_{x=-r},$$

or, observing that $[x+n-1]^n = [x+n-1]^{n-r-1}(x+r)\,[x+r-1]^r$,

$$A_r = \left\{\frac{[x-1]^n}{[x+n-1]^{n-r-1}\,[x+r-1]^r}\right\}_{x=-r} = \frac{[-r-1]^n}{[n-r-1]^{n-r-1}\,[-1]^r}$$

$$= \frac{(-)^n\,[n+r]^n}{[n-r-1]^{n-r-1}\,(-)^r\,[r]^r} = (-)^{n+r}\frac{[n+r]^{n+r}}{[n-r-1]^{n-r-1}\,[r]^r\,[r]^r}$$

$$= (-)^{n+r}\,\frac{[n+r]^{2r+1}}{[r]^r\,.\,[r]^r}.$$

So that the formula is

$$\frac{[x-1]^n}{[x+n-1]^n} = 1+(-)^n\,.\,S_r(-)^r\,.\,\frac{[n+r]^{2r+1}}{[r]^r\,[r]^r}\,\frac{1}{x+r},$$

or, as this may also be written, $\dfrac{(x-1)\,(x-2)\,..\,(x-n)}{x\,(x+1)\,..\,(x+n-1)} =$

$$1+(-)^n\left\{n\,.\,\frac{1}{x}-\frac{n\,(n^2-1^2)}{1^2}\,.\,\frac{1}{x+1}+\frac{n\,(n^2-1^2)\,(n^2-2^2)}{1^2\,.\,2^2}\,.\,\frac{1}{x+2}-\&c.\right\}$$

which is the formula in question.

Note by the Proposer.—To test the restriction in (2), let $n=\frac{1}{2}$; then $x=\frac{1}{2}$ ought to give a correct result, and $x=-\frac{1}{2}$ an erroneous one, which is the case.

Thus $x=\frac{1}{2}$ gives $\dfrac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(1)\,\Gamma(0)}\;(=0) = 1-\dfrac{1}{2}-\dfrac{1\,.\,1}{2\,.\,4}-\dfrac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}-\&c. = (1-1)^{\frac{1}{2}}$

and $x=-\frac{1}{2}$ gives $\dfrac{\{\Gamma(-\frac{1}{2})\}^2}{\Gamma(0)\,\Gamma(-1)}\;(=0) = 1+\dfrac{1}{2}+\dfrac{1\,.\,3}{2\,.\,4}+\dfrac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}+\&c.$

By giving to x and n different values in accordance with this restriction, numerous remarkable results may be obtained; thus from $x=1$ and $x=\frac{1}{2}$, n being perfectly general, we obtain

$$\frac{\{\Gamma(1)\}^2}{\Gamma(1-n)\;\Gamma(1+n)} = \frac{\sin n\pi}{n\pi} = 1-\frac{n^2}{1^2}+\frac{n^2\,(n^2-1^2)}{1^2\,.\,2^2}-\frac{n^2(n^2-1^2)\,(n^2-2^2)}{1^2\,.\,2^2\,.\,3^2}+\&c.;$$

$$\frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(\frac{1}{2}-n)\;\Gamma(\frac{1}{2}+n)} = \cos n\pi = 1-\frac{n^2}{1}\,.\,\frac{2}{1}+\frac{n^2\,(n^2-1^2)}{1\,.\,2}\,.\,\frac{2^2}{1\,.\,3}-\&c.$$

1514 (Proposed by J. GRIFFITHS, M.A.)—Let P be the point of intersection of the three perpendiculars, and G the centre of gravity of any triangle ABC; also let l, m, n be the middle points of the sides BC, CA, AB; S_l, S_m, S_n, the circles described upon Al, Bm, Cn as diameters, and S_1, S_2, S_3, the circles circumscribing the triangles PBC, PCA, PAB.

It is required to prove

(α) That the circle which passes through l, m, n passes also through the points of intersection, real or imaginary, of the self-conjugate and circumscribing circles of each of the triangles PBC, PCA, PAB, ABC.

(β) That the six points common to the three pairs of circles S_l, S_1; S_m, S_2; S_n, S_3; lie on another circle Σ.

(γ) That the self-conjugate and circumscribing circles of the triangle, the circle which bisects its sides, the circle upon PG as diameter, the circle Σ, and the director of the maximum ellipse that can be inscribed in the triangle, all pass through the same two points, real or imaginary.

I. *Solution by the* PROPOSER.

1. The trilinear equations of the circumscribing circle (S), the self-conjugate circle (S_s), and the circle (S′) which bisects the sides of the triangle of reference ABC are easily shown to be

$$S \equiv a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0;$$
$$S_s \equiv a\cos A\,.\,\alpha^2 + b\cos B\,.\,\beta^2 + c\cos C\,.\,\gamma^2 = 0;$$
$$S' \equiv a\cos A\,.\,\alpha^2 + b\cos B\,.\,\beta^2 + c\cos C\,.\,\gamma^2 - (a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0;$$

whence we have the relation $S' = S_s - S$; which shows that S′ passes through the two points, real or imaginary, which are common to S_s and S.

2. It may be readily inferred from Art. 1 that S′ passes through the points common to the self-conjugate and circumscribing circles of each of the triangles PBC, PCA, PAB; for we know that it passes through the middle points of the sides of each of them. This part of the theorem, however, admits of an easy independent proof. For consider one of the triangles, PBC for instance; the equations of its circumscribing and self-conjugate circles are

$$S_1 \equiv 2a\cos A\,.\,\alpha^2 - a\beta\gamma - (a\cos C - c\cos A)\,\gamma\alpha - (a\cos B - b\cos A)\,\alpha\beta = 0;$$
$$S'_s \equiv -a\cos A\,.\,\alpha^2 + b\cos B\,.\,\beta^2 + c\cos C\,.\,\gamma^2 - 2c\cos A\,.\,\gamma\alpha - 2b\cos A\,.\,\alpha\beta = 0.$$

Hence $S' = S_1 + S'_s$; which proves the theorem.

3. The equations of S_l, S_m, S_n are (see *Educational Times* for May, 1863, Quest. 1368)

$$S_l \equiv b\cos B\,.\,\beta^2 + c\cos C\,.\,\gamma^2 - a\beta\gamma - (b + c\cos A)\,\gamma\alpha - (c + b\cos A)\,\alpha\beta = 0.$$

with similar expressions for S_m and S_n; hence we have

$$2S_l + S_1 = 2S_m + S_2 = 2S_n + S_3 =$$
$$2\,(a\cos A\,.\,\alpha^2 + b\cos B\,.\,\beta^2 + c\cos C\,.\,\gamma^2) - 3\,(a\beta\gamma + b\gamma\alpha + c\alpha\beta);$$

relations which prove that the six points common to (S_l, S_1); (S_m, S_2); (S_n, S_3) lie on the same circle, given by the equation $\Sigma = 0$, where

$\Sigma = \sin 2A \,.\, \alpha^2 + \sin 2B \,.\, \beta^2 + \sin 2C \,.\, \gamma^2 - 3(\sin A \,.\, \beta\gamma + \sin B \,.\, \gamma\alpha + \sin C \,.\, \alpha\beta)$.

4. To prove (γ) I take the equation

$\sin 2A \,.\, \alpha^2 + \sin 2B \,.\, \beta^2 + \sin 2C \,.\, \gamma^2 - \lambda(\sin A \,.\, \beta\gamma + \sin B \,.\, \gamma\alpha + \sin C \,.\, \alpha\beta) = 0$ (A),

where λ is an indeterminate constant. This equation evidently represents a coaxal system of circles, passing through the two points, real or imaginary, which are common to the self-conjugate and circumscribing circles of the triangle of reference; also its line of centres may be shown to coincide with the line PG. Now if we put $\lambda = 1$, (A) will represent the circle upon PG as diameter; for it may be put in the form

$$(\alpha \sin A - \beta \sin B)(\alpha \cos A - \beta \cos B) + (\beta \sin B - \gamma \sin C)(\beta \cos B - \gamma \cos C)$$
$$+ (\gamma \sin C - \alpha \sin A)(\gamma \cos C - \alpha \cos A) = 0,$$

which shows that it passes through the points, (sec A, sec B, sec C), and (cosec A, cosec B, cosec C), or, in other words, through P and G.

If $\lambda = 2$, (A) represents the nine-point circle.

If $\lambda = 3$, (A) represents the circle Σ, as shown in Art. 3.

If $\lambda = 4$, (A) represents the director of the ellipse which touches the sides of the triangle at their middle points. This equation may be readily deduced from the general expression given in Ferrers's *Trilinear Coordinates*, p. 87; or in the *Messenger of Mathematics*, vol. ii. p. 42.

5. The common radical axis of the system represented by (A) is given by the equation

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0.$$

It is evident that this line is the polar of the point G with respect to the self-conjugate circle of the triangle of reference; it is also the axis of homology of the triangles ABC, DEF, where D, E, F are the feet of the perpendiculars from A, B, C on BC, CA, AB.

6. The equations of the radical axes of (S', S_1); (S', S_2) (S', S_3) are

$$-3\alpha \cos A + \beta \cos B + \gamma \cos C = 0;\ \alpha \cos A - 3\beta \cos B + \gamma \cos C = 0;$$
$$\alpha \cos A + \beta \cos B - 3\gamma \cos C = 0.$$

The centres of the self-conjugate circles of the triangles PBC, PCA, PAB, coincide with the vertices A, B, C, respectively; and their radii are geometric means between AP, AD; BP, BE; CP, CF. It is evident that these three circles will be *all* real, if the triangle of reference be acute-angled.

It is worthy of notice that the circles in question are also the directors of the conics whose equations are

$$\sqrt{(-a\alpha)} + \sqrt{(b\beta)} + \sqrt{(c\gamma)} = 0;\ \sqrt{(a\alpha)} + \sqrt{(-b\beta)} + \sqrt{(c\gamma)} = 0;$$
$$\sqrt{(a\alpha)} + \sqrt{(b\beta)} + \sqrt{(-c\gamma)} = 0.$$

7. If we take the nine points through which the nine-point circle passes three and three together, we shall get 84 different triangles, to each of which it is the *circumscribed circle*. Therefore, applying the theorem (α), we conclude that the nine point circle of any triangle passes through the points of intersection, real or imaginary, of the self-conjugate circle and the *nine point circle* of each of the above 84 triangles.

II. *Solution by* Mr. W. K. Clifford; *and* F. D. Thomson, B.A.

Let $U \equiv a^2yz + b^2zx + c^2xy$,

$V \equiv (x+y+z)(bc \cos A \,.\, x + ca \cos B \,.\, y + ab \cos C \,.\, z)$;

then $U = kV$ represents respectively (the coordinates being triangular)

(1) when $k = \frac{1}{2}$, the *nine-point* circle of the triangle of reference;

(2) when $k = 0$, the *circumscribing* circle;

(3) when $k = 1$, the *self-conjugate* circle;

(4) when $k = \frac{1}{3}$, the *director of the maximum ellipse*;

(5) when $k = \frac{2}{3}$, the *circle on* PG *as diameter*.

The last two are the only ones which present any difficulty.

In (4) the tangential equations of the ellipse and the circular points at infinity are respectively

$$yz + zx + xy = 0$$

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bc \cos A \,.\, yz - 2ca \cos B \,.\, zx - 2ab \cos C \,.\, xy = 0.$$

Forming, by the rule, the *harmonic conic* of these two, we at once write down equation (4). To find equation (5), we make use of the following

Theorem. If $A=0$, $B=0$, $C=0$, $D=0$ are any four lines in a plane, and if $\Psi AB = 0$ denote the condition that A and B may be at right angles, then, the equation to the circle whose diameter is the line joining (AB), (CD) is

$$\begin{vmatrix} \Psi AC, & \Psi AD, & A \\ \Psi BC, & \Psi BD, & B \\ C, & D, & 0 \end{vmatrix} = 0$$

In the present case, take the four lines $A \equiv x - y$; $B \equiv y - z$;

$C \equiv bc \cos A \,.\, x - ca \cos B \,.\, y$; $D \equiv ca \cos B \,.\, y - ab \cos C \,.\, z$,

then the condition of perpendicularity of $(l_1 m_1 n_1)$ $(l_2 m_2 n_2)$ being

$a^2 l_1 l_2 + b^2 m_1 m_2 + c^2 n_1 n_2 - bc \cos A \,(m_1 n_2 + m_2 n_1) - \&c. = 0$, we have

$\Psi AC = \Psi BD = abc\,(a \cos A + b \cos B + c \cos C) = 2abc \,.\, a \sin B \sin C$,

$\Psi AD = \Psi BC = -abc \,.\, b \sin C \sin A = -\frac{1}{2}\Psi AC$.

Expanding then the determinant, it becomes simply

$$2(AC + BD) + AD + BC = 0,$$

and, substituting the values of A, B, C, D, we get equation (5).

1469. (Proposed by R. Tucker, M.A.)—To investigate some properties of *Radial Curves*. (Continued from the Solution of Question 1397.)

Solution by the Proposer.

24. To find by *Radials* what curves have evolutes similar to themselves.

Let R, R′ be the radii of curvature at corresponding points of the curve

and its evolute; then the angle (α) between them, since the curves are supposed similar, will be constant. Our equations then are

$$R = f(\theta), \quad R' = af(\theta + \alpha), \quad dR = R'd\theta:$$

hence the functional equation is

$$f'(\theta) = af(\theta + \alpha),$$

the general solution of which is (Salmon's *Higher Plane Curves*, p. 281) $R = \Sigma Ce^{m\theta}$, m being one of the roots of the equation $m = ae^{m\alpha}$, and a particular solution

$$R = Ce^{m\theta} \cos n\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A),$$

or the curves whose radials are comprised under the general form (A) have their evolutes similar to themselves.

Their equations in rectangular coordinates must be found (Salmon's Higher Curves, Art. 302) by eliminating θ from the equations

$$x = \int Ce^{m\theta} \cos n\theta \cos \theta \, d\theta, \quad y = \int Ce^{m\theta} \cos n\theta \sin \theta \, d\theta,$$

which are easily integrated, but it does not appear that the elimination can be effected in the general case.

25. The equation (A) assumes very simple forms for particular values of m and n; thus (1) make $n = 0$, and (2) make $m = 0$, then

$$R = Ce^{m\theta} \ldots\ldots (1); \quad R = C \cos n\theta \ldots\ldots (2).$$

By a reference to Art. 17, we see that (1) gives the equiangular spiral as a particular case. This is given as the *only* solution by Gregory (*Examples*, pp. 456, 457), who refers to a memoir by *Euler* on the subject. Equation (2) gives an Epicycloid or Hypocycloid, according as n is less or greater than unity. (See Art. 17.) For further information on the subject, see *Gregory*, and also several articles (302—304) in *Salmon*, as above.

26. We next proceed to find the *Radials* for some given curves.

The curve $xy^2 = a^3$ gives $\cot \theta' = \dfrac{dy}{dx} = -\dfrac{y^3}{2a^3}$ and $\dfrac{d^2y}{dx^2} = -\dfrac{3y^2}{2a^3} \cot \theta'$,

therefore by (β) its Radial is $3r \sin^{\frac{4}{3}} \theta \cos^{\frac{5}{3}} \theta = -2^{\frac{1}{3}} a$.

27. For the *Cardioid* $r = a\,(1 - \cos \theta)$, we have $\theta' = \frac{3}{2}\theta$, $r' = \frac{4}{3}a \sin \frac{1}{3}\theta$, hence by ($\beta$) its Radial is $r = \frac{4}{3}a \sin \frac{1}{3}\theta$, and (by Art. 17) its intrinsic equation is $s = 4a\,(1 - \cos \frac{1}{3}\theta)$

28. We may here notice that our θ' is not always equal to the ϕ of Art. 9, (as in the above example, Art. 27), but sometimes differs from it by a right angle. Regard must in all cases be paid to the definition given in the works referred to in Art. 9.

29. The curve $\dfrac{x}{ae} = e^{\frac{y}{x}}$ gives $\dfrac{dy}{dx} = \log \dfrac{x}{a}$, and $\dfrac{d^2y}{dx^2} = \dfrac{1}{x}$;

therefore by (β) its *Radial* is $r \sin^3 \theta = ae^{\cot \theta}$.

30. To determine the curve whose Radial is the hyperbola $xy = \frac{1}{4}c^2$ we have

$$r^2 \sin \theta \cos \theta = \tfrac{1}{4}c^2, \ \cot \theta = \frac{dy}{dx} = p, \ (r \sin^3 \theta)^{-1} = \frac{d^2y}{dx^2} = q,$$

$$\therefore\ c^2q^2 = 4p\,(1+p^2)^2, \text{ or } \frac{2}{c}\,dx = \frac{dp}{p^{\frac{1}{2}}\,(1+p^2)},$$

$$\text{whence}\quad 2^{\frac{3}{2}}\,\frac{x}{c} = \log\frac{1+\sqrt{(2p)}+p}{1-\sqrt{(2p)}+p} + 2\tan^{-1}\frac{\sqrt{(2p)}}{1-p}\,.$$

Or we might proceed thus :—

$$x = \int r\cos\theta\,d\theta = \tfrac{1}{2}c\int(\cot\theta)^{\frac{1}{2}}\,d\theta,\quad y = \int r\sin\theta\,d\theta = \tfrac{1}{2}c\int(\tan\theta)^{\frac{1}{2}}\,d\theta\,;$$

and the curve obtained by eliminating θ would be the one required.

31. When a given curve has a point of inflexion, since the curve there coincides with a straight line, its radius of curvature at the same point will be infinite, and therefore the Radial *may have* an asymptote. This will be the case if, in the Radial Equation, the polar subtangent is finite.

$$\text{Now the polar subtangent} = r'^2\frac{d\theta'}{dr'} = -\rho\,\frac{ds}{d\rho} = \frac{(1+p^2)^{\frac{3}{2}}\,q}{(1+p^2)\,r-3pq^2}\,\cdots\cdots\cdots(\text{A})$$

hence (1), if $r\ \left(=\dfrac{d^3y}{dx^3}\right)$ be not zero, we have an asymptote passing through the Radial pole at the angle $\cot^{-1}(p)$ with the initial line; and (2) if r be zero, then (A) assumes an indeterminate form, but from Art. 48 below it appears that in this case the Radial has a point of inflexion.

32. If the *sectorial* area of the Radial vary as the *ordinate*, the *primitive* will from (ζ) be given by the differential equation $kpq = -(1+p^2)^2$, k being a constant;

$$\therefore\ p^2 = \left(\frac{dy}{dx}\right)^2 = \frac{k-2x}{2x},\ \text{and}\ 2y = \sqrt{\left\{2x\,(k-2x)\right\}} + k\sin^{-1}\sqrt{\left(\frac{2x}{k}\right)}.$$

If the area vary as the *arc*, we have similarly $kqds = (1+p^2)^2\,dx$, or $k = (1+p^2)^{\frac{3}{2}} \div q = \rho$, that is, the *radius of curvature* of the primitive is *constant*. The primitive is therefore a *circle*, for which curve the property manifestly holds.

33. For the curve $y = a\sin\dfrac{x}{a}$, we have $\dfrac{dy}{dx} = \cot\theta' = \cos\dfrac{x}{a}$,

$\dfrac{d^2y}{dx^2} = -\dfrac{1}{a}\sin\dfrac{x}{a}$, therefore the Radial is $r\sin^2\theta\,(-\cos 2\theta)^{\frac{1}{2}} = -a$.

34. For the curve $y = a\tan\dfrac{x}{a}$ the Radial is

$$2r\sin^2\theta\cos\theta\,(\cot\theta-1)^{\frac{1}{2}} = a,\ \text{or}\ r\sin^{\frac{1}{2}}\theta\,\sin 2\theta\,(1-\sin 2\theta)^{\frac{1}{2}} = a.$$

35. For the curve $y = a\sec\dfrac{x}{a}$ the Radial is

$$r^4\sin^6\theta\cos^2\theta\,(1+3\cos^2\theta)^2 + a^2r^2\sin^4\theta\,(1+3\cos^2\theta) = a^4.$$

36. If the Radial be $r\cos^2\theta = a$, we have $r\sin^3\theta\cot^2\theta = a\sin\theta$,

$$\therefore \text{ by } (\beta),\ \frac{q}{p^2(1+p^2)^{\frac{1}{2}}} = \frac{1}{a}, \text{ whence } p = \frac{dy}{dx} = \frac{a}{(x^2-a^2)^{\frac{1}{2}}},$$

and the *primitive* is the *Catenary*, as might be inferred from Art. 13.

37. Let ρ_1, ρ_2, be the radii of curvature of a curve and its inverse at the points $(r_1\theta)$, $(r_2\theta)$; then $r_1 r_2 = k^2$, and, if $u = \dfrac{1}{r_1}$, $r_2 = k^2 u$,

$$\frac{dr_2}{d\theta} = k^2\frac{du}{d\theta},\ \frac{d^2r_2}{d\theta^2} = k^2\frac{d^2u}{d\theta^2};$$

$$\therefore \rho_2 = \left\{r_2^2 + \left(\frac{dr_2}{d\theta}\right)^2\right\}^{\frac{3}{2}} \div \left\{r_2^2 + 2\left(\frac{dr_2}{d\theta}\right)^2 - r_2\frac{d^2r_2}{d\theta^2}\right\}$$

$$= k^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}} \div \left\{u^2 + 2\left(\frac{du}{d\theta}\right)^2 - u\frac{d^2u}{d\theta^2}\right\}$$

$$\therefore \frac{\rho_2}{\rho_1} = k^2u^3\left(u + \frac{d^2u}{d\theta^2}\right) \div \left\{u^2 + 2\left(\frac{du}{d\theta}\right)^2 - u\frac{d^2u}{d\theta^2}\right\} \ldots\ldots (\eta).$$

38. The cissoid is well known to be the inverse of a parabola, the pole being the vertex of the curve. The equation of the parabola is $r\sin^2\theta = 4a\cos\theta$, and if we take $k^2 = 8a^2$, the equation to the Cissoid will be $r = 2a\sin\theta\tan\theta$, or $(2a-x)\,y^2 = x^3$.

Substituting in (η) the value of u or $\dfrac{\sin^2\theta}{4a\cos\theta}$, we get

$$\frac{\rho_2}{\rho_1} = \frac{\tan^4\theta}{6} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If now θ_1, θ_2 be the angles corresponding to θ' in (a) we have

$$\text{in the parabola, } \tan\theta = \frac{4a}{y} = 2\cot\theta_1 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{in the cissoid, } \cot\theta_2 = \frac{dy}{dx} = \frac{(3a-x)\,x^{\frac{1}{2}}}{(2x-x)^{\frac{3}{2}}} = \frac{3a-x}{2a-x}\,.\tan\theta \ldots (3).$$

From (2) and (3) $\dfrac{\cot\theta_2}{2\cot\theta_1} = \dfrac{3a-x}{2a-x} = 1 + \dfrac{a}{2a-x}$,

$\therefore\ \dfrac{x}{a} = \dfrac{2\cot\theta_2 - 6\cot\theta_1}{\cot\theta_2 - 2\cot\theta_1} = \dfrac{r_2\cos\theta}{a}$, and $r_1\sin\theta = 2a\tan\theta_1$.

From these two equations and $r_1 r_2 = 8a^2$ we obtain

$$4\cot^3\theta_1 + 3\cot\theta_1 - \cot\theta_2 = 0,$$

$$\therefore \cot\theta_1 = \tfrac{1}{2}\left\{\cot^{\frac{1}{3}}\frac{\theta_2}{2} - \tan^{\frac{1}{3}}\frac{\theta_2}{2}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From (1), (2), (4) and Art. 10 we have for the *Radial of the Cissoid*

$$r = \frac{a}{3}\left(\cot^{\frac{1}{3}}\frac{\theta}{2} - \tan^{\frac{1}{3}}\frac{\theta}{2}\right)\left(\cot^{\frac{1}{3}}\frac{\theta}{2} + \tan^{\frac{1}{3}}\frac{\theta}{2}\right)^3$$

39. By means of Art. 17, we can apply the above equation to find the ***intrinsic equation to the Cissoid***, remembering that ϕ and θ are the same; for by (γ)

$$ds = r d\phi = \frac{a}{3}\left(\cot^{\frac{1}{3}}\frac{\phi}{2} - \tan^{\frac{1}{3}}\frac{\phi}{2}\right)\left(\cot^{\frac{1}{3}}\frac{\phi}{2} + \tan^{\frac{1}{3}}\frac{\phi}{2}\right)^3 d\phi.$$

Assume $x = \cot^{\frac{1}{3}}\frac{\phi}{2} + \tan^{\frac{1}{3}}\frac{\phi}{2}$; then

$$\frac{dx}{d\phi} = -\tfrac{1}{6}\left\{\cot^{-\frac{2}{3}}\frac{\phi}{2}\operatorname{cosec}^2\frac{\phi}{2} - \tan^{-\frac{2}{3}}\frac{\phi}{2}\sec^2\frac{\phi}{2}\right\}$$

$$= -\tfrac{1}{6}\left\{\tan^{\frac{2}{3}}\frac{\phi}{2}\left(1+\cot^2\frac{\phi}{2}\right) - \cot^{\frac{2}{3}}\frac{\phi}{2}\left(1+\tan^2\frac{\phi}{2}\right)\right\}$$

$$= +\tfrac{1}{6}\left(\cot^{\frac{2}{3}}\frac{\phi}{2} - \tan^{\frac{2}{3}}\frac{\phi}{2}\right)\left(1 - \cot^{\frac{2}{3}}\frac{\phi}{2} - \tan^{\frac{2}{3}}\frac{\phi}{2}\right)$$

$$= \tfrac{1}{6}x\sqrt{(x^2-4)}\,(1-x^2+2) = -\tfrac{1}{6}(x^2-3)\,x\sqrt{(x^2-4)},$$

$$\therefore\ s = 2a\int\frac{x^2dx}{x^2-3} = 2a\left\{x + \frac{\sqrt{3}}{2}\log\frac{x-\sqrt{3}}{x+\sqrt{3}}\right\}.$$

where x has the value given above.

40. We will here give the Radials for a few curves, the method of obtaining them being readily seen from preceding cases.

The first of the following pairs is the *primitive*, the second its ***Radial.***

(1) $y^m = a^{m-1}x$, $r\sin^3\theta\,(m\cot\theta)^{\frac{2m-1}{m-1}} = -\dfrac{m^2a}{m-1}$;

(2) $y^2 = mx + nx^2$, $2r(\cos^2\theta - n\sin^2\theta)^{\frac{3}{2}} = -m$;

(3) $\left(\dfrac{x}{a}\right)^m \pm \left(\dfrac{y}{b}\right)^n = 1$,

$$\frac{1}{r\sin^3\theta} = -(n-1)\frac{b}{a^2}\left(\mp\frac{a\cot\theta}{b}\right)^{\frac{n-2}{n-1}}\left\{\left(\mp\frac{a\cot\theta}{b}\right)^{\frac{n}{n-1}} \pm 1\right\}^{\frac{n+1}{n}};$$

(4) $x^3 = ay(x-b)$; $a(\cot\theta)\,p^2q = b(p+q)^2(2p-q)$,
where $p^3 = 2r\sin^3\theta$, $q^3 = a - 2r\sin^3\theta$;

(5) $xy^2 = 4a^2(2a-x)$, $4\tan^2\theta = p^3(2-p)$,
where $p^2 = \frac{8}{9} \pm \left(\dfrac{4a}{r}\sec^3\theta - 4\tan^2\theta + \frac{9}{4}\right)^{\frac{1}{3}}$;

(6) $x^3 - 3axy + y^3 = 0$, $2aq^{\frac{1}{3}}(1 + q\cot\theta) = p^{\frac{1}{3}}(q^2 + \cot\theta)$,

where $q^3 = \dfrac{3a + 2r(\sin^3\theta + \cos^3\theta)}{3a - 2r(\sin^3\theta + \cos^3\theta)}$, $p = 9a^2 - 4r^2(\sin^3\theta + \cos^3\theta)^2$;

(7) $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$, $r = -3a \cos 2\theta$;

(8) $e^{\frac{y}{a}} = \tan \frac{x}{c}$, $r \sin 2\theta \,(c^2 \cos^2 \theta - 4a^2 \sin^2 \theta) = \pm\, 2ac$;

(9) $ye^{-y} = x$, $p\,(1+\log p) = \tan\theta$,

where $p = (r \cos^3\theta)^{-1} - \tan\theta$;

(10) For $r^n = a^n \sin n\theta$, we have

$$\phi = \tan^{-1} r \frac{d\theta}{dr} = n\theta, \quad r' = \frac{r \operatorname{cosec} n\theta}{1+n},$$

therefore the Radial is $r = \dfrac{a}{1+n}\left(\operatorname{cosec} \dfrac{n\theta}{1+n}\right)^{\frac{n-1}{n}}$;

(11) For $r^n = a^n = \sec n\theta$ we have

$$\phi = \tfrac{1}{2}\pi - n\theta, \quad r' = \frac{r \sec n\theta}{1-n},$$

therefore the Radial is $r = \dfrac{a}{1-n}\left(\sec \dfrac{n}{1-n}\right)^{\frac{n+1}{n}}$.

41. We may prove, as before, that the curve whose Radial is $r = a \sin 2\theta$ is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (\frac{2}{3}a)^{\frac{2}{3}}$. This can be readily verified from the converse case (3) in Art. 40.

42. From the Radial $r \sin 2\theta = a$ we have

$\dfrac{dy}{dx} = \dfrac{2}{e^{-\frac{2x}{a}} - e^{\frac{2x}{a}}}$, and thence the equation to the primitive is

$$e^{\frac{2y}{a}} = \frac{e^{-\frac{x}{a}} - e^{\frac{x}{a}}}{e^{-\frac{x}{a}} + e^{\frac{x}{a}}}.$$

43. We here append a few more Radials :—

(1) For $x+y = a \tan \frac{x}{c}$, we have $\frac{c}{a}\,(1+\cot\theta) - 1 =$

$\dfrac{c^2}{4\,(1+\sin 2\theta)\, r^2 \sin^4\theta}$, which, if $a = c$, becomes

$$r^2 \sin^2\theta \sin 2\theta\,(1+\sin 2\theta) = \tfrac{1}{2}c^2;$$

(2) for $x+y = a \log \frac{x}{c}$, we have $r \sin\theta\,(1+\sin 2\theta) = -a$;

(3) for $x+y = ae^{\frac{x}{c}}$, we have $r \sin^2\theta\,(1+\sin 2\theta)^{\frac{1}{2}} = c$;

(4) for $x+y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$, we have $r^2\sin^4\theta\,(1+\sin 2\theta+\sin^2\theta)=c^2$;

(5) for $y=e^{\tan x}$, we have $\cot\theta=e^{p}(1+p^2)$,
where $p=(r\sin^2\theta\cos\theta)^{-\frac{1}{2}}-1$;

(6) for $y=e^{\frac{x-y}{y}}$, we have $-r\cos^3\theta=e^{\tan\theta-2}$;

(7) for $y=e^{e^{x}}$, we have $\cot\theta=pe^{p}$, where
$p=(r\sin^2\theta\cos\theta)^{-1}-1$;

(8) for $\sin\frac{y}{a}=\frac{x}{a}$ we have $r\cos^2\theta\,(\cos^{\frac{1}{2}}2\theta)=a$;

(9) for $\tan\frac{y}{a}=\frac{x}{a}$ we have $r\sin 2\theta\cos^{\frac{1}{2}}\theta\,(1-\sin 2\theta)^{\frac{1}{2}}=-a$.

Compare these last two results with Arts. 33, 34, from which they are at once found by changing θ into $\frac{1}{2}\pi-\theta$.

44. Let (x,y), (x',y') be corresponding points on a curve and its Radial, then by (a) we have

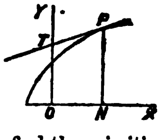
$$x'^2+y'^2=r'^2,\quad \frac{x'}{y'}=\cot\theta'=\frac{dy}{dx};\quad \therefore\ \frac{d^2y}{dx^2}=\frac{y'-p'x'}{y'^2}\frac{dx'}{dx}.$$

$$\text{But } r'\frac{d^2y}{dx^2}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}=\operatorname{cosec}^3\theta=\left(\frac{r'}{y'}\right)^3;$$

$$\left.\begin{aligned}\therefore\ \frac{dx}{dx'}&=\frac{y'(y-p'x')}{x'^2+y'^2},\quad \frac{dy}{dx'}=\frac{x'(y'-p'x')}{x'^2+y'^2}\\ \text{also}\ \frac{ds}{dx'}&=\frac{y'-p'x'}{r'},\quad\text{and}\quad \frac{ds}{dy'}=\frac{y'-p'x'}{p'r'}\end{aligned}\right\}\ \ldots\ldots\ldots(\theta).$$

45. If ON, PN, be the coordinates of a point P on the Radial referred to axes OX, OY, and OT the intercept on OY made by the tangent at P, the equations in Art. 44 may be thus written:—

$$\frac{dx}{dx'}=\frac{\text{PN}\,.\,\text{OT}}{(\text{OP})^2},\quad \frac{dy}{dx'}=\frac{\text{ON}\,.\,\text{OT}}{(\text{OP})^2},\quad \frac{ds}{dx'}=\frac{\text{OT}}{\text{OP}}.$$



46. The equations (θ) in Art. 44 may be applied to find the primitive when the Radial is given. For example, let the Radial be the curve

$$y'=ax'^{n},\ \text{then}\ \frac{dy'}{dx'}=nax'^{n-1}$$

$$\therefore\ dx=a^2(1-n)\frac{x'^{2n-2}\,dx'}{1+a^2x'^{2n-2}},\quad dy=a(1-n)\frac{x'^{n-1}\,dx'}{1+a^2x'^{2n-}};$$

we must then find, from these equations (when the integration can be effected), x and y in terms of x', and the equation which results from the elimination of x' gives the primitive curve.

47. Representing the differential coefficients of y by p, q, r, s, &c., we have

$$y' = (1+p^2) + q, \quad x' = (1+p^2)\,p + q,$$

$$\therefore\ \frac{dy'}{dx'} = p' = \frac{dy'}{dx} \div \frac{dx'}{dx} = \frac{2pq^2-(1+p^2)\,r}{(1+3p^2)\,q^2-(1+p^2)\,rp} \ldots\ldots\ldots\ldots (\iota)$$

$$\frac{d^2y'}{dx'^2} = \frac{q^3\left\{2q^4\,(1-3p^2)-(1+p^2)^2\,(qs-r^2)+4pq^2r\,(1+p^2\right\}}{\left\{(1+3p^2)\,q^2-(1+p^2)\,rp\right\}^3} \ldots (\kappa)$$

$$\rho' = \frac{(1+p^2)^{\frac{3}{2}}\left\{q^4+[3pq^2-r\,(1+p^2)]^2\right\}^{\frac{3}{2}}}{q^3\left\{2q^4\,(1-3p^2)+4pq^2r\,(1+p^2)-(qs-r^2)\,(1+p^2)^2\right\}} \ldots (\lambda)$$

48. From the equations (ι), (κ) many interesting properties may be derived. Thus, if a curve and its Radial have the same inclination at corresponding points, we have by (ι)

$$(1-3p^2)\,pq^2 = (1-p^4)\,r, \quad \text{whence} \quad e^{b-y\sqrt{2}} - e^{-(b-y\sqrt{2})} = 2\sin(a-x\sqrt{2}).$$

Again, from (κ), if the Radial have a point of inflexion, q may be zero, hence (Art. 31) instead of an asymptote only for the Radial, we may have in a particular case a point of inflexion corresponding to a point of inflexion on the primitive; but in this case, by (ι), we have $p' = p^{-1}$, which gives us the curve (A) just found, the axes being turned through a right angle. Hence, generally, if a Radial have a point of inflexion, the relation $q = 0$, or

$$2q^4\,(1-3p^2)+4pq^2r\,(1+p^2)-(qs-r^2)\,(1+p^2)^2 = 0,$$

must hold for the primitive.

49. We may also apply the equations in Art. 47 to find the condition for the existence of an asymptote for the Radial parallel to the axis of y'. From the value of y', q must be zero, and from that of x', p and q must contain the vanishing factor; hence, from (ι), p' is ∞, and there will be an asymptote for a point of inflexion on the primitive.

50. If ϕ' be the angle between the radius vector at the point (r', θ') of a radial and the tangent at that point, and s, s', s'' the corresponding lengths of arc for a curve, its evolute, and its radial, we have, by (γ) and Todhunter's *Differential Calculus*, § 310, 331,

$$\sin\phi' = r'\frac{d\theta'}{ds''} = -\frac{ds}{ds''}, \quad \cos\phi' = \frac{dr'}{ds''} = -\frac{ds'}{ds''};$$

$$\therefore (ds'')^2 = (ds)^2 + (ds')^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mu).$$

$$\text{Or thus,} \quad \left(\frac{ds''}{d\theta}\right)^2 = r'^2 + \left(\frac{dr'}{d\theta}\right)^2 = \left(\frac{ds}{d\theta}\right)^2 + \left(\frac{ds'}{d\theta}\right)^2.$$

51. Let $r^2 = a^2\cos 2\theta$ be the equation to the Lemniscate; then, referring to the Solution of Quest. 1407, we have, for the Radial of the *Evolute*,

$$\cot\theta' = \frac{dy}{dx} = \tan 3\theta, \quad \therefore\ 3\theta = \tfrac{1}{2}\pi - \theta',$$

and $r' \sin^3\theta' = \left(\dfrac{d^2y}{dx^2}\right)^{-1} = -\dfrac{a \sin 2\theta \cos^3 3\theta}{9(\cos 2\theta)^{\frac{3}{2}}} = -\dfrac{a \sin(\frac{1}{3}\pi - \frac{2}{3}\theta') \sin^3\theta'}{9\left\{\cos(\frac{1}{3}\pi - \frac{2}{3}\theta')\right\}^{\frac{3}{2}}}$;

$\therefore\ 9r \cos^{\frac{1}{2}}(\frac{1}{3}\pi - \frac{2}{3}\theta) = -a \tan(\frac{1}{3}\pi - \frac{2}{3}\theta)$ is the Radial to this Evolute.

52. Referring to Gregory's *Examples*, p. 197, we have for the differential equation to the Evolute of the Catenary,

$$2c\frac{d\beta}{d\alpha} = -\left\{(\alpha+c)^2 - 4c^2\right\}^{\frac{1}{2}}, \quad \therefore\ 2c\frac{d^2\beta}{d\alpha^2} = \frac{(\alpha+c)}{2c\cot\theta'};$$

$$\text{hence} \quad (\alpha + c) = \frac{4c^2\cot\theta'}{r'\sin^3\theta'} = 2c \operatorname{cosec}\theta',$$

$\therefore\ r\sin^3\theta = 2c\cos\theta$ is the Radial of this Evolute.

53. The curve whose Radial is $r\cos 3\theta = a$ is found, by the usual substitutions, from $\sqrt{(1 - e^{6a^{-1}x})}\,dy = \sqrt{3}\,.\,dx$, whence we obtain

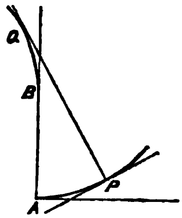
$$e^{\frac{y\sqrt{3}}{a}} + e^{-\frac{y\sqrt{3}}{a}} = 2e^{-\frac{3x}{a}}.$$

Similarly, from the Radial $r\cos 2\theta\cos\theta = -a$, we get $\dfrac{dx}{dy} = \sqrt{(1 - e^{2a^{-1}x})}$,

whence we obtain $e^{\frac{y}{a}} + e^{-\frac{y}{a}} = 2e^{\frac{x}{a}}$ as the curve required.

54. Let (α, β) be a point on the Evolute corresponding to (x, y) on the primitive; then, by Art. 47 and Todhunter's *Differential Calculus*, § 320, we have $y - \beta = -y'$, $x - \alpha = x'$; $\therefore\ dy - d\beta = -dy'$, $dx - d\alpha = dx'$. Hence, squaring and adding, remembering that $dy\,d\beta = -dx\,d\alpha$, we have $(ds)^2 + (ds')^2 = (ds'')^2$, which furnishes another proof of (μ).

55. Let AP be the primitive and BQ the Evolute, then, if $s = f(\phi)$ be the intrinsic equation of AP, $s' = \dfrac{ds}{d\phi} - c$ is that of BQ (Todhunter's *Int. Calc.*, § 114), and $ks'' = \dfrac{ds}{d\phi} - c$ will be that of the Radial, when the Radial and Evolute are similar,



$\therefore$ by (μ), $\left(\dfrac{ds}{d\phi}\right)^2 + \left(\dfrac{d^2s}{d\phi^2}\right)^2 = \dfrac{1}{k^2}\left(\dfrac{d^2s}{d\phi^2}\right)^2$, or $\dfrac{d^2s}{d\phi^2} = c\dfrac{ds}{d\phi}$;

$$\text{whence } \frac{ds}{d\phi} = be^{c\phi}, \text{ and } s = \frac{b}{c}e^{c\phi},$$

which is the equation to the logarithmic Spiral.

56. From equations (θ) and (μ) we obtain $\dfrac{ds'}{dx'} = \dfrac{x' + p'y'}{r'}$,

$$\therefore\ \frac{ds'}{ds} = \frac{x' + p'y'}{y' - p'x'} = -\tan(\theta' + \theta'') = \cot\phi', \text{ where } \theta'' = \cot^{-1}p',$$

$$\text{and, by } (\mu),\ \frac{ds''}{ds} = \pm\sec(\theta' + \theta'') = \mp\operatorname{cosec}\phi' \ \ldots\ldots\ldots\ldots (\nu.)$$

57. The equations (μ) and (ν) will enable us to find the intrinsic equations of the Radial and the Evolute of the Radial, if we know that of the primitive. Let $s=f(\phi)$ be the equation of the primitive, then, by Art. 55, $\pm s' = \frac{ds}{d\phi} - c$ is that of the Evolute; hence, by (μ) and (ν),

$$\left(\frac{ds''}{d\phi}\right)^2 = \left(\frac{ds}{d\phi}\right)^2 + \left(\frac{d^2s}{d\phi^2}\right)^2 \dots\dots\dots\dots\dots\dots\text{(i.)}$$

$$\left(\frac{ds''}{ds}\right)^2 = \sec^2(\phi + \phi_1) \dots\dots\dots\dots\dots\dots\text{(ii.)},$$

where ϕ_1 is the angle for the radial corresponding to the angle ϕ of the primitive (Art. 9). From (i.) and (ii.) and $s = f(\phi)$, we can determine $\frac{ds''}{d\phi_1}$ in terms of ϕ_1, and hence obtain the equations of the Radial and its Evolute.

58. As an application of the equations in Art. 57, let us take the equiangular Spiral, whose equation is $s = ae^{k\phi}$, then

$$\frac{ds}{d\phi} = ake^{k\phi}, \quad \text{and} \quad \frac{d^2s}{d\phi^2} = ak^2e^{k\phi};$$

hence, substituting in (i.) and (ii.), we get $\tan(\phi+\phi_1) = k$ and $d\phi = -d\phi_1$;

$$\therefore \frac{ds''}{d\phi_1} = be^{-k\phi_1} \quad \text{and} \quad s'' = ce^{-k\phi_1};$$

hence, as in Art. 17, the Radial is also an equiangular Spiral.

Again, for the Cycloid, we have

$$s = 4a \sin\phi, \quad \frac{ds}{d\phi} = 4a\cos\phi, \quad \frac{d^2s}{d\phi^2} = -4a\sin\phi;$$

$$\therefore \frac{ds''}{d\phi} = 4a = 4a\sec(\phi+\phi_1)\cos\phi; \quad \text{hence} \quad \phi_1 = -2\phi \text{ or } 2\pi - 2\phi,$$

$$\text{and } \frac{ds''}{d\phi_1} = -2a; \quad \therefore s'' = -2a\phi_1,$$

the equation to a circle, radius $2a$ (see Art. 7).

59. We will here give one or two results, the cases being worked out as in the last Article.

For the curve $s = c\sec\phi \dots\dots\dots\dots\dots\dots\dots\dots$ (A),

$$\text{we get} \quad \tan\phi\tan\phi_1 = \tfrac{1}{2}, \quad \text{and} \quad \frac{ds''}{d\phi_1} = \frac{c}{2}\operatorname{cosec}^3\phi_1;$$

which is (Art. 17) the equation to a parabola ($x^2 = cy$); hence, by Art. 21, we see that (A) is the intrinsic equation of the curve

$$\frac{x}{c} = \left(e^{\frac{2y}{c}} - 1\right)^{\frac{1}{2}} - \tan^{-1}\left(e^{\frac{2y}{c}} - 1\right)^{\frac{1}{2}}.$$

For the catenary $s = c \tan\phi$ we get

$$\frac{ds''}{d\phi_1} = \frac{c \sec^2\phi_1 (1 + 2\tan^2\phi)^2 (1 + 4\tan^2\phi)^{\frac{1}{2}}}{1 - 2\tan^2\phi} \ldots\ldots\ldots\ldots\ldots\ldots \text{(i.)}$$

$$\text{and } \tan\phi_1 = \tan\phi \div (1 + 2\tan^2\phi) \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii.)},$$

whence, eliminating ϕ, we get the equation to the Evolute of the Radial, and, by integration, that of the Radial.

60. Returning to our equations (μ) and (ν), we have, denoting by R the radius of curvature of the Evolute of the primitive,

$$\rho' \frac{d\phi_1}{d\phi} = \sqrt{(\rho^2 + \mathrm{R}^2)} = \rho \sec(\phi + \phi_1) \ldots\ldots\ldots\ldots\ldots\ldots (\xi);$$

$$\text{hence } \mathrm{R} = \rho \tan(\phi + \phi_1) = \rho \cot\phi' = -\rho \frac{d\rho}{ds},$$

as can be readily shown otherwise (see Art. 5).

61. For an equiangular Spiral we have $d\phi = -d\phi_1$, (see Art. 58), hence, by (ξ), the property $\rho'^2 = \rho^2 + \mathrm{R}^2$ holds for that curve. Similarly for the Cycloid we have $4\rho'^2 = \rho^2 + \mathrm{R}^2$.

62. If we adopt a more symmetrical notation, we have, denoting the 1st, 2nd,...... nth Radials by the equations

$$s_1 = f_1(\phi_1), \quad s_2 = f_2(\phi_2), \ \ldots\ldots \ s_n = f_n(\phi_n),$$

$$\frac{ds_1}{ds} = \sec(\phi + \phi_1), \quad \frac{ds_2}{ds_1} = \sec(\phi_1 + \phi_2), \ \ldots\ldots \ \frac{ds_n}{ds_{n-1}} = \sec(\phi_{n-1} + \phi_n);$$

$$\therefore \frac{ds_n}{ds} = \sec(\phi + \phi_1)\sec(\phi_1 + \phi_2) \ldots\ldots \sec(\phi_{n-1} + \phi_n).$$

From these equations, taken in connexion with the equations formed as in (μ), we may find the nth Radial, when we know the intrinsic equation to the primitive.

63. Again, by (μ), we have

$$(ds_1)^2 = (ds)^2 + (d^2s)^2, \ (ds_2)^2 = (ds_1)^2 + (d^2s_1)^2, \ \ldots \ (ds_n)^2 = (ds_{n-1})^2 + (d^2s_{n-1})^2;$$

$$\therefore (ds_n)^2 = (ds)^2 + (d^2s)^2 + (d^2s_1)^2 + \ldots + (d^2s_{n-1})^2,$$

$$\text{and } \rho_n^2 \left(\frac{d\phi_n}{d\phi}\right)^2 = \rho^2 + \mathrm{R}^2 + \mathrm{R}_1^2 \left(\frac{d\phi_1}{d\phi}\right)^2 + \ldots\ldots + \mathrm{R}_{n-1}^2 \left(\frac{d\phi_{n-1}}{d\phi}\right)^2 \ldots\ldots (\pi),$$

where by ρ_n, R_n we mean the radii of curvature of the nth Radial and its Evolute respectively. Now from (ξ) we get

$$\rho_1^2 \left(\frac{d\phi_1}{d\phi}\right)^2 = \rho^2 + \mathrm{R}^2, \quad \rho_2^2 \left(\frac{d\phi_2}{d\phi_1}\right)^2 = \rho_1^2 + \mathrm{R}_1^2, \quad \&c.$$

$$\therefore \rho_1^2 \rho_2^2 \ldots\ldots \rho_n^2 \left(\frac{d\phi_n}{d\phi}\right)^2 = (\rho^2 + \mathrm{R}^2)(\rho_1^2 + \mathrm{R}_1^2) \ldots (\rho_{n-1}^2 + \mathrm{R}_{n-1}^2) \ldots\ldots (\sigma).$$

By means of the equations (π) or (σ) we can express ρ_n in terms of radii of

lower orders; for $\frac{d\phi_1}{d\phi}$, $\frac{d\phi_2}{d\phi_1}$, ... $\frac{d\phi_n}{d\phi_{n-1}}$ can be found as in Arts. 58, 59. The equation (σ) is merely the equation of Art. 62 in a different form.

64. Since the polar subtangent of the Radial $= \frac{r^2 d\theta}{dr} = -\left(\frac{ds}{d\phi}\right)^2 \div \frac{d^2 s}{d\phi^2}$, we have, for the intrinsic equation to the primitive of the Radial $r\theta = a$,

$$\left(\frac{ds}{d\phi}\right)^2 = a\left(\frac{d^2 s}{d\phi^2}\right), \quad \therefore s = a \log \frac{1}{\phi}, \text{ and } \phi = e^{-\frac{s}{a}}.$$

65. The polar equation to the locus of the end of the polar subtangent of the Radial is given by the equations

$$\theta = \phi, \quad r = -\left(\frac{ds}{d\phi}\right)^2 \div \frac{d^2 s}{d\phi^2}.$$

This locus for the Cycloid is $r = a \cot\theta \cos\theta$, for the Catenary $r \sin 2\theta = a$, and for the equiangular Spiral $r = -ae^{k\theta}$.

1482. (Proposed by the Rev. R. H. Wright, M.A.)—Find the equation to a straight line which is perpendicular to a given line $l\alpha + m\beta + n\gamma = 0$ and bisects the portion of it intercepted by the sides β and γ of the triangle of reference.

Solution by Alpha; *and* Mr. A. Renshaw.

The equation of a line through A (or $\beta\gamma$) parallel to the given line (l, m, n) is

$$(m \sin A - l \sin B)\,\beta + (n \sin A - l \sin C)\,\gamma = 0 \quad \text{............} \quad (1),$$

and the harmonic conjugate of (1) with respect to β and γ, which will bisect the part of (l, m, n) between β and γ, is

$$(m \sin A - l \sin B)\,\beta - (n \sin A - l \sin C)\,\gamma = 0 \quad \text{............} \quad (2).$$

The equation of a line through the intersection of (2) with (l, m, n) is

$$(m \sin A - l \sin B)\,\beta - (n \sin A - l \sin C)\,\gamma + k\,(l\alpha + m\beta + n\gamma) = 0 \text{......} \quad (3);$$

and applying the condition that (3) should be perpendicular to (l, m, n), we find

$$k = \frac{(n^2 - m^2)\sin A - l^2 \sin(B-C) + 2lm \sin B - 2ln \sin C}{l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C},$$

and thus the required equation (3) becomes

$$l\left\{l^2 \sin (B-C) + (m^2-n^2) \sin A - 2lm \sin B + 2ln \sin C\right\}\alpha$$

$$+ \left\{l^2 (l \sin B - 2m \sin A - n \sin 2B) + m^2 [l \sin (A-C) + n \sin 2A] + n^2 (l \sin B - 2m \sin A) + 4lmn \sin A \cos B\right\}\beta$$

$$- \left\{l^2 (l \sin C - 2n \sin A - m \sin 2C) + m^2 (l \sin C - 2n \sin A) + n^2 [l \sin (A-B) + m \sin 2A] + 4lmn \sin A \cos C\right\}\gamma = 0.$$

1483 (Proposed by Mr. J. WILSON.)—Given two circles, one within the other, then

(α.) Another circle may be found, through the centre of which if any straight line be drawn cutting the other two, the segments within the ring will subtend equal angles at any point on the circumference of the latter.

(β.) Two points may be found, through either of which if any straight line be drawn cutting the given circles, the segments within the ring shall subtend equal angles at the other.

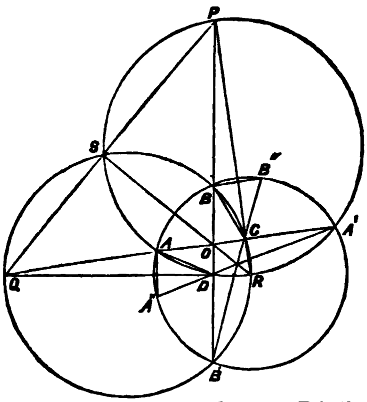
(γ.) Two points may be found such that the external segments of any straight line cutting the circles shall subtend equal angles at them.

Solution by the REV. R. TOWNSEND, M.A.; *and the* PROPOSER.

The two points E and F (Cor. 2°, Art. 155, *Modern Geometry*), inverse to each other with respect to both circles, that is the two limiting points of the coaxal system they determine, (Art. 184, *Modern Geometry*), are the points in (β) and (γ), (Arts. 163, Cor. 1°, and 192, Cor. 4°, *Modern Geometry*); and *every* circle passing through them—as cutting the two given circles orthogonally—possesses the property (α). For if O be its centre, P and Q, P′ and Q′, the pairs of intersections with the given circles of any line through O, then OP . OQ = OP′ . OQ′ = square of radius of orthogonal circle, therefore (α) follows by Art. 161, *Modern Geometry*.

1484 (Proposed by MATTHEW COLLINS, B.A.)—If two chords of a circle, AA′ and BB′, cut at O, and if OB and OB′ subtend equal angles at C, the middle of AA′; prove conversely that OA and OA′ must also subtend equal angles at D, the middle of BB′; also prove that CB + CB′ = DA + DA′.

Solution by J. M. WILSON, M.A.; *and* Mr. A. RENSHAW.



Bisect BB′ at right angles by the line QR, meeting CA, and the perpendicular to CA through C, in the points Q, R. On QR as diameter describe a circle, which will obviously pass through B and B′, since $\angle$ BCQ = B′CQ (see McDowell's *Exercises*, Prop. 77). Produce BB′ and RC to meet in P; join PQ; draw ROS perpendicular to PQ; and on PR as diameter describe a circle passing through S, D.

Then OA . OA′ = OB . OB′
= OR . OS;

therefore *a* circle passes through S, A, R, A′, and its centre must be on RP, because this line bisects AA′ at right angles; also R is one end of a diameter, and RSP is a right angle, therefore P is the other end, and *the* circle PSR passes through A, A′;

$$\therefore \ \angle ADB = A'DB.$$

Again, produce B′C, A′D to meet the circle in B″, A″; then CB, CB″ are equally inclined to the diameter CP, and are therefore equal; also BB″ is parallel to AA′; and the like for AA″;

$$\therefore \ \angle B'BB'' = A'OB' = A'AA'';$$

hence the chords A′A″, B′B″ are equal, that is

$$CB + CB' = DA + DA'.$$

1486 (Proposed by Mr. W. K. CLIFFORD.)—If two transversals ABC, DEF cut the sides of any triangle, then AE, BF, CD cut the sides in three points on a straight line X. If moreover the triangle touch a cubic at A, B, C, and cut it in D, E, F, the lines AE, BF, CD meet the curve in three points on a straight line Y, and the lines X, Y, ABC, meet in a point.

Solution by MR. W. H. H. HUDSON.

Let the equations of the transversals referred to the given triangle PQR be $l\alpha + m\beta + n\gamma = 0$ (ABC) and $l'\alpha + m'\beta + n'\gamma = 0$ (DEF). The equation of a cubic which touches the sides of the triangle along A, B, C, and cuts them along D, E, F, is

$$\alpha\beta\gamma + (l\alpha + m\beta)^2 (l'\alpha + m'\beta) + (m\beta + n\gamma)^2 (m'\beta + n'\gamma) + (n\gamma + l\alpha)^2 (n'\gamma + l'\alpha)$$
$$- l^2l'\alpha^3 - m^2m'\beta^3 - n^2n'\gamma^3 \equiv \phi(\alpha, \beta, \gamma) = 0.$$

The equation of PA is $m\beta + n\gamma = 0$, and of QE is $n'\gamma + l'\alpha = 0$;
therefore the equation of AE is $nl'\alpha + mn'\beta + nn'\gamma \equiv w = 0$:
similarly the equation of BF is $ll'\alpha + lm'\beta + nl'\gamma \equiv u = 0$;
and the equation of CD is $lm'\alpha + mm'\beta + mn'\gamma \equiv v = 0$.

The equation of the straight line X, passing through (u, α), (v, β), (w, γ), is

$$ll'm'n\alpha + mm'n'l\beta + nn'l'm\gamma \equiv \xi = 0 \ldots\ldots\ldots\ldots\ldots\ldots (X).$$

Now since ABC, DEF are straight lines in a cubic, AE, BF, CD intersect the cubic in a straight line Y or $\lambda\alpha + \mu\beta + \nu\gamma = 0$; consequently if k be a properly determined constant

$$uvw + (l\alpha + m\beta + n\gamma)(l'\alpha + m'\beta + n'\gamma)(\lambda\alpha + \mu\beta + \nu\gamma) \equiv k\,\phi(\alpha, \beta, \gamma).$$

Equating coefficients of α^3 we find $\lambda = kl - ll'm'n$; and similarly μ, ν may be found from the coefficients of β^3, γ^3; hence the equation of Y is

$$k(l\alpha + m\beta + n\gamma) - \xi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (Y).$$

Therefore X, Y, ABC meet in a point.

1488 (Proposed by JOHN CASEY, B.A.)—If two lines be divided homographically, prove by Elementary Geometry that

(α.) Two points can be found, such that, if perpendiculars be drawn from them on the lines joining homologous points, the feet of all the perpendiculars are in the circumference of a circle.

(β.) If the lines joining two pairs of homologous points be perpendicular to each other, the locus of their intersection is a circle.

(γ.) This circle cuts orthogonally the polar circles of the triangles formed by every three lines of the system.

Solution by the PROPOSER.

1. Let L, L′ be the two lines divided homographically in a, b, c, &c., and a', b', c', &c. (Fig. 1); a, a'; b, b'; c, c'; &c. being the homologous points; and let P be the point on the line L which corresponds to infinity on L′, and P′ the point on L′ which corresponds to infinity on L, then $Pa \cdot P'a' = Pb \cdot P'b' = Pc \cdot P'c'$, &c. = a *constant* = k suppose (Chasles's *Géometrie Supérieure*, Art. 120). Construct the triangle PFP′ having the rectangle of the sides PF . FP′ = k, and the difference of the angles at the base equal to the difference of the angles LPP′ and L′P′P. (This is the well-known problem, to con-

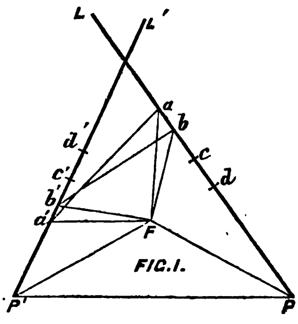


FIG. 1.

struct a triangle having given the base, the difference of the angles at the base, and the rectangle of the sides.) Join Fa, Fa'; Fb, Fb'. Now, by construction $PF \cdot P'F = Pa \cdot P'a'$, $\therefore FP' : P'a' = Pa : PF$, and $\angle a'P'F = aPF$, hence $\angle FaP = a'FP'$; and in like manner $\angle FbP = b'FP'$, $\therefore \angle aFb = a'Fb'$, hence $\angle aFa' = bFb'$, that is, the line joining homologous points subtends a constant angle at the point F.

2. Again, let aa' (Fig. 2) be a variable line intersecting L, L′ in a, a', and subtending a constant angle at F. Draw the perpendiculars FA, FB, FC, and let A′, B′, C′, be the poles of aa', L, L′, with respect to a circle of given radius and centre F. Join A′B′, A′C′; then A′B′ is the polar of a', hence Fa' cuts A′B′ perpendicularly; and in like manner Fa is perpendicular to A′C′, therefore the angle B′A′C′ is the supplement of aFa', and is given. Also the points B′, C′, being the poles of L, L′, are given, hence the locus of A′ is a circle. And since FA . FA′ is given, the locus of A is a circle inverse to the former. Moreover FA is a radius vector to a given circle from a given point, therefore the perpendicular to Aa at the point where it meets the circle again passes through another given point (F′) equidistant from, and in the same line with, the centre and F. Hence if two lines be divided homographically, we have found two F, F′, which points satisfy the conditions in (a).

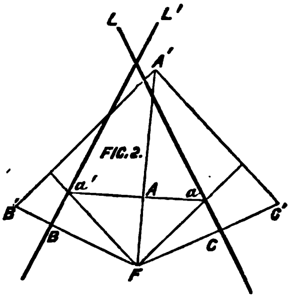


FIG. 2.

3. Let aa', bb', (Fig. 3) be two lines of the system, intersecting perpendicularly in M; and let F be one of the points determined in Art. 1, and O the centre of the circle which passes through the feet of the perpendiculars from F. Then since AFBM is a rectangle, $OF^2 + OM^2 = OA^2 + OB^2$, therefore OM is given and the locus of M is a circle, which proves (β).

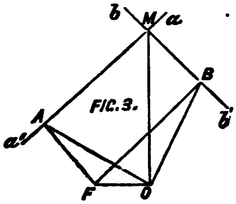


FIG. 3.

4. To prove (γ), we observe that the polar circles of the four triangles formed by any four lines of the system are coaxal, (see Townsend's *Modern Geometry*, Art. 189); and their radical axis passes through the centre of the circle through the feet of the perpendiculars; hence the polar circles of the triangles formed by all the lines of the system have a common orthogonal circle, which, by supposing two lines of the system to intersect perpendicularly, is evidently the circle determined in Art. 3.

1489 (Proposed by HUGH GODFRAY, M.A.)—Two particles, of weights P and Q, are connected by a fine inextensible thread of length a. Q rests

on a smooth table, and P is just over the edge. It is known from previous experiments that the thread snaps when Q is stopped suddenly, being still on the table, after having been drawn a distance equal to or greater than b.

Show that, if $a > \left(\frac{P+Q}{Q}\right)^2 b$, the thread will break when Q leaves the table, and determine the subsequent path of Q.

If $a < \left(\frac{P+Q}{Q}\right)^2 b$, show that, after Q has left the table, the direction of its motion will at certain instants be vertical, provided $Q < P$; and determine its positions at those instants.

Solution by the PROPOSER.

1. The moving force on the system is the weight P, and the mass moved is the sum of the masses of P and Q, therefore the acceleration f is $\frac{P}{P+Q}g$; also the velocity acquired in describing a space x is $\sqrt{(2fx)}$, and if T′ be the impulsive tension of the string when Q is suddenly stopped after describing a space x, this tension must destroy P's velocity,

$$\therefore\ T' = \frac{P}{g}\,(\text{vel.}) = \frac{P}{g}\sqrt{(2fx)}.$$

Let T be the jerk required to break the string; then $T' = T$ when $x = b$,

$$\therefore\ T = \frac{P}{g}\sqrt{(2fb)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)}$$

2. When Q reaches the end of the table, its velocity horizontally outwards will be $\sqrt{(2fa)}$; and P will have the same velocity vertically downwards. The tension of the string upon Q, which had hitherto been horizontal, becomes suddenly vertical. The instantaneous velocity of the centre of gravity of the now free system is

$$\frac{Q}{P+Q}\sqrt{(2fa)}\ \text{horizontally}\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)};$$

$$\frac{P}{P+Q}\sqrt{(2fa)}\ \text{vertically downwards}\ldots\ldots\ldots\ldots\text{(iii.)}$$

Therefore, if T″ be the impulsive tension of the string, T″ acting downwards must give this latter velocity to Q,

$$\therefore\ T'' = \frac{Q}{g}\cdot\frac{P}{P+Q}\sqrt{(2fa)}\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv.)}$$

Comparing (iv.) with (i.), we see that T″ will be greater than T, and therefore the string will break

$$\text{if } \frac{Q}{P+Q} > \sqrt{\left(\frac{b}{a}\right)},\ \text{ or if } a > \left(\frac{P+Q}{Q}\right)^2 b.$$

3. Suppose $a > \left(\frac{P+Q}{Q}\right)^2 b$, and therefore the string to break; then the initial velocity of Q will be $\sqrt{(2fa)}$ horizontally, and $\frac{Tg}{Q} = \frac{P}{Q}\sqrt{(2fb)}$ vertically downwards; and from these data we may determine the parabolic path.

4. Suppose $a < \left(\frac{P+Q}{Q}\right)^2 b$, and consequently the string *not* to break; then (ii.) and (iii.) give the initial velocity of the centre of gravity. Let x, y be its horizontal and vertical coordinates (y downwards) at any subsequent time t, and θ the angle which the string then makes with the vertical; then

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = g, \quad \frac{d^2\theta}{dt^2} = 0; \quad \text{whence, by integration,}$$

$$\left.\begin{aligned} \frac{dx}{dt} &= \text{constant} = \frac{Q}{P+Q}\sqrt{(2fa)}, \text{ by (ii.)} \\ \frac{dy}{dt} &= gt + \text{const.} = gt + \frac{P}{P+Q}\sqrt{(2fa)}, \text{ by (iii.)} \\ \frac{d\theta}{dt} &= \text{constant} = \frac{\text{Q's initial hor. vel.}}{a} = \sqrt{\left(\frac{2f}{a}\right)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (\text{v.})$$

and, integrating once more, we shall obtain x, y, θ in terms of t.

5. If x', y' be the coordinates of Q at time t, the distance of Q from the centre of gravity being $\frac{P}{P+Q}a$, we have

$$x' = x + \frac{P}{P+Q}a\sin\theta, \quad y' = y - \frac{P}{P+Q}a\cos\theta \ldots\ldots\ldots\ldots (\text{vi.})$$

$$\therefore \quad \frac{dx'}{dt} = \frac{dx}{dt} + \frac{P}{P+Q}a\cos\theta\frac{d\theta}{dt} = \frac{Q+P\cos\theta}{P+Q}\sqrt{(2fa)}.$$

In order that Q's motion may be vertical, $\frac{dx'}{dt}$ must $=0$, whence $\cos\theta = -\frac{Q}{P}$, a possible value of θ when $Q < P$.

6. From (v.) we obtain $\theta = \sqrt{\left(\frac{2f}{a}\right)}t$, no constant being added, since θ and t begin together; whence $t = \sqrt{\left(\frac{a}{2f}\right)}\cos^{-1}\left(-\frac{Q}{P}\right)$.

This equation gives the times when the motion of Q is vertical, and (v.) will by integration furnish the corresponding values of x and y; and the positions of Q at the same time will, by (vi.), be given by

$$x' = x \pm a\sqrt{\left(\frac{P-Q}{P+Q}\right)}, \quad y' = y + a\frac{Q}{P+Q}.$$

1506 (Proposed by R. Tucker, M.A.)—Find the envelope of a series of circles which pass through a fixed point, and intercept a given length on a fixed straight line.

Solution by DR. HIRST, F.R.S.

It may readily be shown that through the given point O and any assumed point M two, and only two, circles can be drawn so as to intercept upon the given line (L) a segment ab of constant length. In fact, if OM cut (L) in m, the segments ma, mb are determined in magnitude, since their sum (or difference when O and M are on the same side of L) is given and their rectangle is equal to $mO \cdot mM$. The *position* of a (or b) to the right or left of m alone remains arbitrary, whence it follows that (1) *through any assumed point pass in general two of the enveloping circles.* In order that these two circles may coincide, and M become a point of the envelope, it is obvious that the segments ma, mb must be equal; whence we conclude that (2) *the required envelope* (E) *is also the locus of the extremity of a produced radius vector of the given line, the rectangle under the vector and its production being equal to the square on half the given intercepted segment.* This property enables us to trace the curve with facility. It consists of a single branch situated wholly on the side of (L) opposite to O; it is symmetrical with respect to the perpendicular on (L) from O, in crossing which orthogonally it reaches its greatest distance from L; it has two points of inflexion and approaches (L) asymptotically.

If we invert the whole figure with respect to the point O, we learn at once from (1) that the inverse of the envelope (E) is a curve (E_1) of the *second class*, and that this conic (E_1), which is of course symmetrical with respect to the perpendicular from O on (L), passes through the origin O (since E has a real point at infinity); in other words, that O is a vertex of (E_1). And as (E) has no real branches passing through O, we at once conclude that (3) *the required envelope is the inverse, with respect to O, of an ellipse* (E_1) *whose vertex is at O, and whose axes are perpendicular and parallel to* (L). As such, it follows at once from the theory of inverse curves (see Quest. 1471) that (4) *the required envelope* (E) *is a circular cubic having a conjugate point at O, and three pointic contact at infinity with the line* (L). The inverse of (L) is the circle of curvature of the ellipse (E_1) which, as is well known, has four pointic contact at the vertex O. The two circles which pass through O and osculate the ellipse (E_1) elsewhere give, by inversion, the stationary tangents of (E); and by a general law the foci of the ellipse give, by inversion, the foci of the cubic. We may add that a similar method of analysis leads easily to the solution of the more general problem proposed as Question 1560.

[NOTE.—Several correspondents have sent algebraical solutions of this Question, but as the methods are simple and obvious, it will be sufficient to give the results. Taking the fixed point O as origin of rectangular axes, and the perpendicular (c) from it on the fixed line (L) as axis of x, and putting $2\lambda c$ for the segment intercepted on (L) by the circle, the equation of the required envelope (E) is

$$(x^2+y^2)(x-c)=\lambda^2cx^2, \quad \text{or} \quad r\cos\theta=c\,(1+\lambda^2\cos^2\theta)\ldots\ldots\ldots\ldots(\text{E}).$$

We may add that the inverse (E_1) of (E) with respect to a circle of radius μc around O is

$$(1+\lambda^2\cos^2\theta)\,r=\mu^2c\cos\theta, \quad \text{or} \quad (1+\lambda^2)\,x^2+y^2-\mu^2cx=0\ldots\ldots(\text{E}_1),$$

an ellipse of which O is a vertex at the end of the *minor* axis; a result agreeing with Dr. Hirst's (3), since the osculating circle of (E_1) at O, which is the inverse of (L), must clearly lie *outside* the ellipse. Putting $\lambda=\tan\alpha$, the major axis parallel to (L) is $\mu^2c\cos\alpha$, the minor axis perpendicular to (L) is $\mu^2c\cos^2\alpha$, and the eccentricity is $\sin\alpha$.

Mr. Tucker remarks that the locus of the centres of the variable circles is a *parabola*, which gives the following property of a parabola:—" A point O is taken on the axis at a distance from the vertex equal to the semi-parameter; a circle described with any point P on the curve as centre, and radius PO, intercepts on the tangent at the vertex a segment equal to the parameter."

The locus of the centres of the variable circles is, of course, similar to that of the points diametrically opposite to the *origin*, on these circles; and the latter is clearly the first negative pedal of (E), that is to say, the reciprocal of (E_1), (see Quest. 1442). Since (E_1) passes through the origin, however, this reciprocal is obviously a *parabola*.—EDITOR.]

1508 (Proposed by the Rev. R. TOWNSEND, M.A.)—Prove the following general properties of inverse figures:—

(α) Every two figures E and F, inverse to each other with respect to a circle C, invert from any point O into two figures E′ and F′, inverse to each other with respect to the inverse circle C′.

(β) When the centre of inversion O is on the circle C, they invert into two similar, equal, and opposite figures, reflexions of each other with respect to the line inverse to C.

Solution by ARCHER STANLEY; MR. J. WILSON; *and the* PROPOSER.

It will obviously suffice to show that the theorem (α) is true when E, F, E′, F′ are simply points, and this may readily be done by means of the following theorems (see Townsend's *Modern Geometry*, Chap. ix.):—

1. Every circle through a pair of inverse points cuts the circle of inversion orthogonally; and conversely

2 Two points will be inverse to each other, with respect to a circle, if every circle passing through them cut the fixed circle orthogonally.

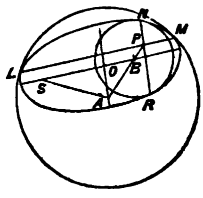
3. The angles formed, at corresponding intersections, by any two curves and by their inverse curves, are similar.

Now E and F being inverse points with respect to (C), every circle through the former cuts the latter orthogonally by (1), and hence every circle through E′ and F′ must by (3) cut C′ orthogonally; whence it follows by (2) that E′ and F′ are inverse points with respect to (C′).

Remembering that the radius of (C′) becomes infinite when O is on (C), the theorem (β) is at once recognized to be a special case of (α).

1516 (Proposed by M. W. CROFTON, B.A.)—Find the locus (1) of the centre, and (2) of the foci, of an ellipse which has double contact with each of two given circles, one being within the other.

Solution by F. D. THOMSON, M.A.; MR. J. WILSON; *and the* PROPOSER.



1. Let A, B be the centres of the circles, and L, M, N, R the points of contact. By a well-known property of the ellipse, the common chords of an ellipse and a circle which cut it are equally inclined to the axis; and hence, when a circle has double contact with an ellipse, the common chord is parallel to the major or minor axis according as the contact is external or internal. Hence in the figure LM is parallel to the major axis, and NR to the minor axis. Through A and B draw perpendiculars to the two chords, then the point of intersection O is the centre of the ellipse. And since A and B are fixed points, and AOB is a right angle, the locus of O is a circle on AB as diameter.

2. Any point in LM has the same polar with respect to the ellipse and larger circle, and any point in NR the same polar with respect to the ellipse and smaller circle; hence P, the intersection of the chords LM, NR, has the same polar with respect to both circles, and is therefore a *fixed* point. Now a, b being the semi-axes of the ellipse, we have $a^2 : b^2 =$ LP . PM : NP . PR; hence $a : b$ is given, and therefore the eccentricity (e). But if k be the radius of the larger circle, the distance (AS) of the focus (S) from the centre of that circle is equal to ek; hence the locus of the foci is a circle (of radius ek) concentric with the larger circle.

[NOTE.—That AS$=ek$ is equivalent to the following theorem. The length (k) of the normal to an ellipse, from the curve to the point where it meets the *minor* axis, is e^{-1} (or $a : c$) times the distance of this point from the focus. This can be readily proved from Taylor's *Conics*, p. 60, or Salmon's *Conics*, p. 164. Moreover the *fixed point* P is obviously one of the two points inverse to each other with respect to both circles, and may, from this property, be easily determined. See Townsend's ***Modern Geometry***, Art. 188, Cor. 1° and 2°; Art. 155, Cor. 2°.—EDITOR.]

1519 (Proposed by F. D. THOMSON, M.A.)—ABC is a triangle having the three real points (P, Q, R) of inflexion of a cubic on the sides BC, CA, AB respectively, each of which also passes through two imaginary points of inflexion. The tangents at Q and R meet in D, those at R and P in E, and those at P and Q in F. Show that AD, BE, CF meet in a point which is fixed for all the cubics having the same nine points of inflexion.

I. *Solution by* Professor CREMONA, of Bologna.

The connectors AD, BE, CF are concurrent, since, by construction, ABC and DEF are coaxial triangles. Further, these connectors are the respective *harmonic polars* (see Salmon's *Higher Plane Curves*, p. 140, or Cremona's *Teoria Geometrica delle Curve Piane*, p. 113) of P, Q, R, not only with respect

to the cubic whose stationary tangents are EFP, FDQ, DER, but with respect to *every* cubic having the same nine points of inflexion; hence the invariability of their point of concurrence. The latter, in fact, is the pole of the line PQR with respect to every cubic of the pencil, since the harmonic polars are constituent parts of the polar conics of P, Q, R.

II. *Solution by* Mr. W. K. Clifford; Mr. J. Wilson; *and* the Proposer.

Taking ABC for triangle of reference, the equation of the cubic is

$$a^3x^3 + b^3y^3 + c^3z^3 - 3dxyz = 0,$$

which may also be written in the form

$$\left(\frac{d}{bc}x + by + cz\right)\left(\frac{d}{bc}x + \theta by + \theta^2 cz\right)\left(\frac{d}{bc}x + \theta^2 by + \theta cz\right) = \frac{d^3 - a^3b^3c^3}{b^3c^3}x^3,$$

(where θ is an imaginary cube root of unity) showing that the tangents at P, Q, R are

$$\frac{d}{bc}x + by + cz = 0, \quad ax + \frac{d}{ca}y + cz = 0, \quad ax + by + \frac{d}{ab}z = 0.$$

The equation to AD is therefore $by = cz$, so that AD, BE, CF meet in the point $ax = by = cz$.

Now all the points of inflexion are on the axes $xyz = 0$; consequently any other cubic having the same points of inflexion can only differ from the above in the coefficient of xyz, of which the point $ax = by = cz$ is independent. This point is the pole of the line PQR with respect to the triangle ABC.

1523 (Proposed by Alpha.)—If $[x]^n$ denote the factorial

$$x(x-1)(x-2)\ldots\ldots..(x-n+1),$$

show how to interpret $[\pm x]^0$ and $[\pm x]^{-n}$.

Solution by Mr. S. Bills.

Adopting the notation in the Question, we have obviously

$$[x]^n = (x-n+1)[x]^{n-1}, \; \therefore \; [x]^{n-1} = [x]^n \div (x-n+1).$$

Now, since the series $[x]^n$, $[x]^{n-1}$, &c., is to be continuous, let us trace it through $[x]^0$. Commencing, say, at $[x]^4$, we have

$$[x]^4 = x(x-1)(x-2)(x-3); \quad [x]^3 = \frac{[x]^4}{x-3} = x(x-1)(x-2);$$

$$[x]^2 = \frac{[x]^3}{x-2} = x(x-1); \qquad [x]^1 = \frac{[x]^2}{x-1} = x;$$

$$[x]^0 = \frac{[x]^1}{x} = \frac{x}{x} = 1; \qquad [x]^{-1} = \frac{[x]^0}{x+1} = \frac{1}{[x+1]^1};$$

$$[x]^{-2} = \frac{[x]^{-1}}{x+2} = \frac{1}{[x+2]^2}; \qquad [x]^{-3} = \frac{[x]^{-2}}{x+3} = \frac{1}{[x+3]^3}.$$

Proceeding thus, we find

$$[x]^{-n} = \frac{1}{[x+n]^n}.$$

Changing x into $-x$, we have

$$[-x]^{-n} = \frac{(-1)^n}{[x-1]^n}; \quad \text{also,} \quad [-x]^0 = 1.$$

1531 (Proposed by Dr. BOOTH, F.R.S.)—"The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles." What is the more general theorem in the conic sections, of which this is a particular case?

Solution by the PROPOSER.

If a quadrilateral ABCD be *circumscribed* to a conic, it may easily be shown that the opposite sides subtend two right angles at a focus, or AD and CB subtend two right angles at F. Now let us take the polar reciprocal of this property. Let the centre of the conic—an ellipse say—be the centre of the directrix circle; then the polar of the ellipse will be another concentric ellipse whose major and minor axes will coincide with the minor and major axes of the former, the polar reciprocal of the focus of the original ellipse will be the "minor directrix" of the derived ellipse, and the reciprocal of the circumscribed quadrilateral to the original will be an inscribed quadrilateral in the derived ellipse. And as the polar of the point A is a side of the inscribed quadrilateral, and the polar of the point F the minor directrix of the derived conic section, the polar of the line AF will be the point α in which the side of the inscribed quadrilateral cuts the minor directrix. We may show the same for B. Hence $\alpha\beta$ subtends at the centre an angle equal to AFB. Hence the points $\alpha\beta$, $\gamma\delta$, on the minor directrix, subtend two right angles at the centre.

When the conic becomes a circle the minor directrices vanish to infinity; the central radii therefore become parallel to the sides of the inscribed quadrilateral; and as the angles between the former are *always* equal to two right angles, the angles between the latter will ultimately become equal to two right angles, that is, when the *minor directrices* are infinitely distant, or the curve a circle.

In the same way we may take any *focal* or *directrix property* of a conic, and deduce by the geometrical method of *Polars* the reciprocal property with reference to the *minor directrix* or *minor focus* of the reciprocal curve. In the equilateral hyperbola the major and minor directrices coincide, as also

do the major and minor foci; hence the focal properties of the equilateral hyperbola are dual. Thus, if any two chords be drawn from a point in an equilateral hyperbola to the ends of the axis, they will intercept on the directrix a segment which subtends a right angle at the centre; and this in virtue of its being the minor as well as the major directrix of the curve.

Hence also, if from any point on a conic perpendiculars be drawn to its minor directrices, the rectangle under these perpendiculars will have a constant ratio to the square of the semi-diameter passing through the point; and it may be easily shown that this property holds for the common directrix of the equilateral hyperbola.

1501 (Proposed by F. D. Thomson, M.A.)—ABC is a triangle inscribed in a conic; chords AA′, BB′, CC′ are drawn through any point O. Show that if P be any point on the conic, PA′, PB′, PC′ intersect BC, CA, AB respectively in points which are in a straight line. Hence deduce the property of a triangle circumscribing a parabola, that the three perpendiculars meet on the directrix.

Solution by the Rev. R. Townsend, M.A.

Let X, Y, Z be the three points of intersection of PA′, PB′, PC′ with BC, CA, AB respectively; then for the three inscribed hexagons whose vertices in consecutive order are ACC′PB′B, BAA′PC′C, CBB′PA′A, the three triads of points YOZ, ZOX, XOY are those determining their three Pascal lines respectively, therefore the three points X, Y, Z are collinear with each other and with the centre of perspective O of the two triangles ABC and A′B′C′.

If the conic be a circle and the point O its centre, the second part of the Question is the *reciprocal* of the first to *any circle having its centre at* P.

If P′ be the other end of the chord from P through O, and X′, Y′, Z′ the three points of intersection of P′A, P′B, P′C, with B′C′, C′A′, A′B′ respectively; the three points X′, Y′, Z′ are collinear and in involution with the three X, Y, Z. Hence we have the two following theorems, reciprocals of each other. When two triangles, either inscribed or circumscribed to the same conic, are in perspective with each other,

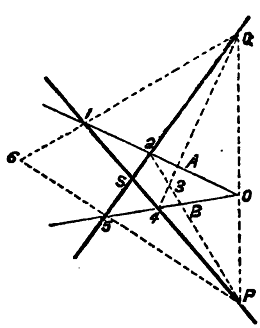
a. Every line passing through their centre of perspective intersects with their three pairs of corresponding sides at three pairs of conjugates of a row in involution.

a′. Every point lying on their axis of perspective connects with their three pairs of corresponding vertices by three pairs of conjugates of a pencil in involution.

Properties which are also evident from the consideration that the centre and axis of perspective, in either case, being pole and polar to each other with respect to the conic, the three pairs of corresponding intersections or connectors, X and X′, Y and Y′, Z and Z′, have a common segment, or angle, of harmonic section, viz., that determined on the line, or at the point, by the centre and axis of perspective.

1512 (Proposed by Professor CAYLEY.)—It is possible to construct a hexagon 123456, inscribed in a conic, and such that the diagonals 14, 25, 36 pass respectively through the Pascalian points (intersections of opposite sides) 23, 56; 34, 61; 45, 12. Given the points 1, 2; 4, 5; to construct the hexagon.

Solution by the PROPOSER.



Let 12, 45, meet in O, and through O draw at pleasure a line meeting 14 in P, and 25 in Q; let P2, Q4 meet in 3, and P5, Q1 in 6; then the line 36 will pass through O, and this being so, the hexagon 123456 satisfies the required conditions.

We have to show that 36 passes through O. Let Q4 meet O12 in A, and P2 meet O45 in B; then the points 6, 3, O, are the intersections of corresponding sides of the triangles A1Q, B5P; and in order that these points may lie in a line, the lines joining the corresponding vertices must meet in a point, that is, we have to show that the lines 15, AB, PQ meet in a point. The property is in fact as follows; viz., given the points 2, 4; and also the points Q, O, P lying in a line; then constructing the points 1, 5, A, B, which are the respective intersections of P4, O2; Q2, O4; Q4, O2; P2, O4; the lines 15, AB, PQ will meet in a point. Take $x = 0$, $y = 0$, $z = 0$ for the respective equations of P2, Q4, PQ; then O is an arbitrary point in the line PQ, say that for the point O we have $z=0$, $ax + by = 0$; also O2, O4 are arbitrary lines through O: say that their equations are $ax + by + \lambda z = 0$; $ax + by + \mu z = 0$; then we have for the points A and B, respectively, $ax + by + \mu z = 0$, $y = 0$; $ax + by + \mu z = 0$, $x = 0$; hence the equation of AB is $\mu ax + \lambda by + \lambda\mu z = 0$. The equation of P4 is $ax + \mu z = 0$, and that of Q2 is $by + \lambda z = 0$; the point 1 is therefore given by $ax + \mu z = 0$, $ax + by + \lambda z = 0$; and 5 by $by + \lambda z = 0$, $ax + by + \mu z = 0$; hence the equation of 15 is $\mu ax + \lambda by + (\mu^2 - \mu\lambda + \lambda^2) z = 0$; and the equation of PQ being $z = 0$, it is clear that the three lines AB, 15, PQ intersect in the point given by the equations $\mu ax + \lambda by = 0$, $z = 0$.

OBS. 1. By inspection of the figure we see that 3PQ is a triangle whereof the sides 3P, 3Q, PQ pass respectively through the fixed points 2, 4, O; while the vertices P and Q lie in the fixed lines 14, 25; the locus of the vertex 3 is consequently a conic; and the like as regards the triangle 6PQ.

OBS. 2. The regular hexagon projects into a hexagon inscribed in a conic and circumscribed about another conic having double contact therewith; in the hexagon so obtained (as appears at once by the consideration of the regular hexagon) the above mentioned property holds; but the in-and-circumscribed hexagon has the additional property that the three diagonals meet in a point, and it is therefore a less general figure than the hexagon of the foregoing theorem. It would, I think, be worth while to study further the hexagon of the theorem.

[NOTE.—In the solution of Question 1548 it is shown that if two pairs of opposite sides of *any hexagon* intersect each on a diagonal produced, so likewise will the third pair.

A slight variation of Professor Cayley's proof may be obtained by finding the equations of P5, Q1, and thence of 36, which are respectively

$$ax-(\lambda-\mu)\,z=0,\ bx+(\lambda-\mu)\,z=0,\ ax+by=0,$$

showing that 36 passes through O.—EDITOR.]

1532 (Proposed by Professor SYLVESTER.)—Show that a law of density for points in space may be assumed such that the joint mass of any two points which are *electrical images* of each other in respect to a given sphere may be constant, and that their centre of gravity shall lie on the surface of the sphere.

Solution by ARCHER STANLEY.

The density at any point in space will clearly be a function, say $F(r)$, of its distance r from the centre of the sphere. The electrical image of any point, at a distance r from the centre, is on the same radius vector and at a distance a^2r^{-1}, where a is the radius of the given sphere; hence, if m be the constant joint mass of any two such points, we have the following two conditions for determining the unknown function F,

$$F(r)+F\left(\frac{a^2}{r}\right)=m,\qquad rF(r)+\frac{a^2}{r}F\left(\frac{a^2}{r}\right)=am;$$

from which we at once deduce $F(r)=\dfrac{am}{a+r}$.

Points at infinity, therefore, will have the density $F(\infty)=0$, those on the sphere the density $F(a)=\frac{1}{2}m$, and the centre of the sphere the density $F(0)=m$.

1534 (Proposed by the Rev. ROBERT HARLEY, F.R.S.)—Given the area of a quadrilateral, one of its sides, and the difference between the sum of the squares on the sides adjacent to the given one and the square on the side opposite to it; show that the locus of the point of intersection of the diagonals is a circle passing through the ends of the given side.

Solution by ARCHER STANLEY; J. GRIFFITHS, M.A.; MR. S. BILLS; *and* MR. J. WILSON.

If $abcd$ be one of the quadrilaterals on the given side ab, and m the intersection of its diagonals ac and bd, then the values of the two expressions,

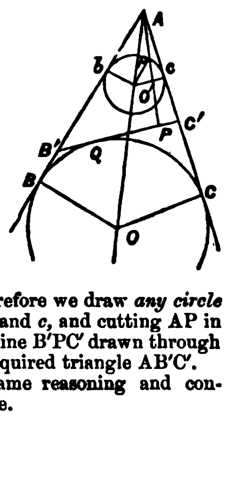
$$(ma \,.\, mb + mb \,.\, mc + mc \,.\, md + md \,.\, ma) \sin amb, \text{ and}$$

$$2(ma \,.\, mb + mb \,.\, mc + mc \,.\, md + md \,.\, ma) \cos amb,$$

are given; since the first represents double the given area, and the second is the difference between the given quantities $bc^2 + ad^2 - cd^2$ and ab^2. By division, therefore, we conclude that tan amb, and hence the angle at m, has always the same magnitude, so that m necessarily lies on a circle passing through a and b.

1535 (Proposed by the EDITOR.)—Through a given point, within a given angle, to draw (*geometrically*) a straight line which shall form, with the lines containing the given angle, a triangle whose perimeter is a *minimum*.

Solution by MATTHEW COLLINS, B.A.; REV. R. TOWNSEND, M.A.; J. M. WILSON, M.A.; *and* G. T. SADLER, F.R.A.S.



Let BAC be the given angle, P the given point within it, and O the centre of a circle touching AB, AC at B and C; then *any* straight line B′QC′ touching this circle in the part convex towards A will cut off a triangle AB′C′ of constant perimeter (=2AB), which perimeter has a given ratio to the radius BO. Hence, to solve the question proposed, we have only to find the smallest circle touching AB and AC to the convex part (towards C) of which a tangent can be drawn from P; that is to say, the circumference must just pass through P. We shall then have OP=OB=OC; and therefore also $O'p = O'b = O'c$, O′ being *any* point in OA, and $O'b$, $O'c$, $O'p$ parallel respectively to OB, OC, OP. If therefore we draw *any circle* $O'pbc$ touching the sides of the given angle at b and c, and cutting AP in two points, of which p is nearest to A, the straight line B′PC′ drawn through P perpendicular to $O'p$ will obviously cut off the required triangle AB′C′.

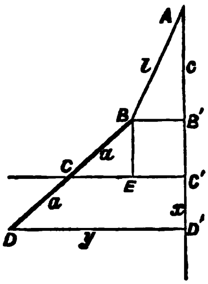
With one or two obvious modifications, the same reasoning and construction apply to the sphere as well as to the plane.

1528 (Proposed by H. J. PURKISS, B.A.)—One end of a uniform rod of length $2a$ is fastened by a string of length l to a fixed point, and the other end rests on a smooth fixed curve in the same vertical plane with the rod and

string; show that, if the curve be such that there is equilibrium when the rod is in contact with any point of it, its equation may be written

$$16y^2(a^2-x^2) = (y^2-3x^2-2cx+4a^2-l^2+c^2)^2.$$

Solution by MATTHEW COLLINS, B.A.; *and the* PROPOSER.



Let AB be the string of length l, BC = CD = a. Now since the curve upon which the end D moves is *smooth*, and there is equilibrium in *any position*, therefore by the principle of virtual velocities the centre of gravity C must move along a fixed horizontal straight line CEC′. Let this meet the vertical lines through A and B in C′, E; draw DD′ horizontal, and put AC′ = c, C′D′ = BE = x, D′D = y; then BB′ = $y-2\sqrt{(a^2-x^2)}$, and AB′ = $c-x$; hence the equation of the curve is

$$\left\{y-2\sqrt{(a^2-x^2)}\right\}^2+(c-x)^2 = l^2,$$

which reduces to

$$16y^2(a^2-x^2) = (y^2-3x^2-2cx+4a^2-l^2+c^2)^2.$$

The locus of D would be found by solving a differential equation of the first order, if the curve were *rough* upon which D would always rest in a state bordering upon motion.

1538 (Proposed by Professor SYLVESTER.)—Show that the discriminant of F, where $F = ax^6+15ex^2+6fx+g$, is

$$a^3\left\{g^5a^2+(1000e^3g^3-7500e^2g^2f^2+9375egf^4-3125f^6)\,a \\ +250000e^6g-150000e^5f^2\right\}$$

Solution by MR. S. BILLS; *and* F. D. THOMSON, M.A.

If K be the discriminant of F, we shall get an equation of the form K = 0 by eliminating x between F = 0 and $\dfrac{dF}{dx} = 0$, that is, between

$$ax^6+15ex^2+6fx+g = 0 \ldots\ldots\ldots (1); \quad ax^5+5ex+f = 0 \ldots\ldots\ldots (2).$$

From $\left\{(1)-x(2)\right\}$ and $\left\{6x(2)-(1)\right\}$ we obtain

$$10ex^2+5fx+g = 0 \ldots\ldots\ldots\ldots (3); \quad 5ax^6+15ex^2-g = 0 \ldots\ldots\ldots\ldots (4).$$

From (3), $10ex^2+g = -5fx$; hence, squaring, we have

$$100e^2x^4+(20eg-25f^2)\,x^2+g^2 = 0 = a_1x^4+b_1x^2+c_1, \text{ say } \ldots\ldots\ldots (5).$$

From $\left\{g(4)+(5)\right\}$ we obtain

$$agx^4+20e^2x^2+(7eg-5f^2)=0=a_2x^4+b_2x^2+c_2, \text{ say} \ldots\ldots\ldots\ldots (6).$$

Eliminating x between (5) and (6), we have

$$(a_1c_2-a_2c_1)^2+(b_1c_2-b_2c_1)(b_1a_2-b_2a_1)=0, \text{ that is,}$$

$$(ag^3-700e^3g+500e^2f^2)^2+25(25f^4-55ef^2g+24e^2g^2)(4aeg^2-5af^2g-400e^4)=0,$$

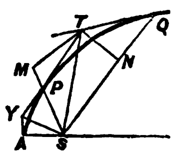
which may be put into the form

$$\text{K}\equiv g^6a^5+(1000e^3g^3-7500e^2f^2g^2+9375ef^4g-3125f^6)a^4 + (250000e^6g-150000e^5f^2)a^3=0.$$

Now it is known that the discriminant of any quantic, say $(a, b, \ldots b', a')(x,y)^m$, contains the term $a^{m-1}a'^{m-1}$, of which the coefficient may be taken equal to unity; hence K, which contains this term, is the required discriminant.

1439 (Proposed by Professor SYLVESTER.)—Prove that the tangents (PT, QT) from any two points (P, Q) of a parabola, as mutually limited by each other, are as their distances from the focus.

Solution by C. TAYLOR, B.A.



If A be the vertex, S the focus, TM perpendicular to SP, and SY to TP, the triangle TPM is similar to SPY or SYA, hence we have

PT : SY = TM : SA; similarly, QT : SZ = TN : SA,

where SZ is perpendicular to TQ, and TN to SQ. But TM, TN are equal, since PT, QT subtend equal angles at S; therefore the tangents PT, QT are as their distances from the focus.

1543 (Proposed by MATTHEW COLLINS, B.A.)—Required a general algebraic expression for the real root of the equation $x^7+7x^5+14x^3+7x=2a$; and show that a class of equations of this kind are solvable algebraically when they are of an odd degree and have only one real root and one general coefficient, viz., the absolute term $2a$.

Solution by the REV. ROBERT HARLEY, F.R.S.

Writing $y-y^{-1}$ in place of x, the equation becomes

$$y^7-y^{-7} = 2a, \text{ whence } y = \omega^m \left\{a \pm (1+a^2)^{\frac{1}{2}}\right\}^{\frac{1}{7}},$$

where ω is any unreal seventh root of unity, and $m = 1, 2, \ldots . 7$ successively. So that *all* the values of x, that is, *all* the roots of the equation, are given by the formula

$$\omega^m \left\{a+(1+a^2)^{\frac{1}{2}}\right\}^{\frac{1}{7}} + \omega^{6m} \left\{a-(1+a^2)^{\frac{1}{2}}\right\}^{\frac{1}{7}}.$$

It is easy to see how to construct equations of this class *ad libitum*. For, let

$$x^{2n+1} + k_1 x^{2n-1} + k_2 x^{2n-3} \ldots . + k_2 x^3 + k_1 x = 2a \quad \ldots\ldots\ldots . (\alpha),$$

and assume as before, $x = y-y^{-1}$. Then we can readily transform the equation in x into another in y, and so determine k_1, k_2, &c., in terms of n as to reduce the equation in y to the solvable form

$$y^{2n+1} - y^{-(2n+1)} = 2a.$$

For the determination of k we have the following system of equations, viz.,

$$k_1-(2n+1) = 0,$$

$$k_2-(2n-1)\,k_1 + \frac{(2n+1)\,2n}{2} = 0,$$

$$k_3-(2n-3)\,k_2 + \frac{(2n-1)\,(2n-2)}{1\,.\,2}\,k_1 - \frac{(2n+1)\,(2n)\,(2n-1)}{1\,.\,2\,.\,3} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$k_m - (2n-2m+3)\,k_{m-1} + \frac{(2n-2m+5)\,(2n-2m+4)}{1\,.\,2}\,k_{m-2}$$

$$+ \ldots\ldots + (-)^m \frac{(2n+1)\,(2n)\,(2n-1)\ldots .(2n-m+2)}{1\,.\,2\,.\,3 \ldots . m} = 0,$$

m being any positive integer. We thus find

$$k_1 = 2n+1,$$

$$2k_2 = (2n+1)\,(2n-2),$$

$$2\,.\,3\;k_3 = (2n+1)\,(2n-3)\,(2n-4),$$

$$2\,.\,3\,.\,4\;k_4 = (2n+1)\,(2n-4)\,(2n-5)\,(2n-6),$$

$$\&c. \qquad \&c. \qquad \&c.$$

$$2\,.\,3 \ldots . m k_m = (2n+1)\,(2n-m)\,(2n-m-1)\ldots .(2n-2m+2).$$

The roots of the equation (α), when these values are assigned to k_1, are all included in the formula

$$\omega \left\{a+(1+a^2)^{\frac{1}{2}}\right\}^{\frac{1}{2n+1}} + \omega^{2n} \left\{a-(1+a^2)^{\frac{1}{2}}\right\}^{\frac{1}{2n+1}},$$

ω representing the $(2n+1)$th roots of unity, each root giving a corresponding value of x.

The foregoing solution suggested a property in the theory of numbers which is proposed as Question 1567, and which may be presented in an extended and generalized form.

1545 (Proposed by F. D. THOMSON, M.A.)—A conic is inscribed in a triangle so that the normals at the points of contact meet in a point: prove that the locus of the centre of the conic is a cubic passing through the angular points of the triangle, the centre of gravity, the intersection of the perpendiculars, the centres of the inscribed and escribed circles, the middle points of the sides and of the three perpendiculars, and the point given by $x : y : z = a^2 : b^2 : c^2$ in areal coordinates.

Solution by DR. HIRST, F.R.S.

I propose to demonstrate, geometrically, the following more general theorem. *The locus of the pole of any line* L *with respect to a conic inscribed in a given triangle* ABC *is a cubic, provided the triangle of contact* $\alpha\beta\gamma$ *be co-polar with a given fixed triangle* DEF. This theorem obviously reduces itself to the given one on supposing D, E, and F to recede to infinity, in directions perpendicular to BC, CA, AB, and the line L to become also infinitely distant.

It is well known that if a series of conics be inscribed in the triangle ABC so as also to touch the line L, the points of contact α, β, γ of the same conic will describe three homographic ranges on BC, CA, AB. Consequently the rays Dα, Eβ, Fγ will describe three homographic pencils, whence we conclude that the locus of the intersection of Dα and Eβ will be a conic through D and E, whilst that of the intersection of Dα and Fγ will be a conic through D and F. Exclusive of D, these two conics intersect in three points which are clearly the only ones in the plane in which the connectors Dα, Eβ, Fγ concur. Hence, of the conics which satisfy the condition expressed in the theorem, there are three which touch the line L. The poles of the line L with respect to these three conics are the only points in which L intersects the required locus; in other words, the latter is a cubic.

That the 16 points referred to in the Question lie in the cubic locus of centres, may be shown thus. (1) Each of the three hyperbolas having two of the sides of the triangle for asymptotes, and touching the third side, evidently fulfils the required conditions; hence their centres, the corners of the triangle ABC, are on the cubic. (2) The sides of the triangle, each considered as a flattened conic, satisfy the required conditions; hence their middle points are on the cubic. (3) The three perpendiculars are also flattened conics satisfying the conditions; hence their middle points are likewise on the cubic. (4) It is well known that the necessary and sufficient condition to be satisfied in order that a conic may touch BC, CA, AB in α, β, γ, is that Aα, Bβ, Cγ should be concurrent. The centre of such a conic is obviously the point of concurrence of the lines through A, B, C, which bisect $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, respectively. (5) It is manifest, therefore, that there is one inscribed conic, whose points of contact bisect the sides of the triangle, and since the perpendiculars in these middle points are concurrent, the conic satisfies the conditions of the question. Its centre, according to the remark in (4), is the centre of gravity G, which accordingly is on the cubic. (6) Another inscribed conic satisfying the conditions is that whose points of contact are the feet of the three perpendiculars. The centre of this conic, constructed according to (4), can be easily shown to be the point whose areal coordinates are proportional to a^2, b^2, c^2. (7) It has already been shown that the cubic locus of centres not only circumscribes the hexagon AGB$\beta'\gamma\alpha'$ (where α, β, γ are the middle points of the sides BC, CA, AB, and α', β', γ' those of the perpendiculars upon these sides), but also passes through the intersections of the two pairs of opposite sides AG, $\gamma\beta'$ and BG, $\gamma\alpha'$; since the latter coincide,

respectively, with the points α and β. The same cubic, therefore, by a well known and easily demonstrated theorem, will also pass through the intersection of the remaining pair of opposite sides, which are obviously the perpendiculars Aα', Bβ'.

The above, together with the centres of the inscribed and circumscribed circles, which are manifestly on the cubic, make up the 16 points alluded to in the Question.

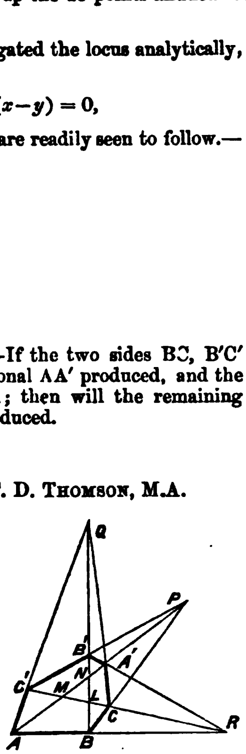
[NOTE.—Several correspondents have investigated the locus analytically, and found its equation in areal coordinates to be

$$a^2yz(y-z)+b^2zx(z-x)+c^2xy(x-y)=0,$$

from which the properties stated in the Question are readily seen to follow.—EDITOR.]

1548. (Proposed by W. A. WHITWORTH.)—If the two sides BC, B′C′ of any hexagon ABCA′B′C′ intersect on the diagonal AA′ produced, and the two sides CA′, C′A on the diagonal BB′ produced; then will the remaining sides AB, A′B′ intersect on the diagonal CC′ produced.

Solutions (I) *by the* PROPOSER; (II) *by* F. D. THOMSON, M.A.



I. Using quadrilinear coordinates, let $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ be the equations of AB, BC, CA′, A′A, where $\alpha+\beta+\gamma+\delta\equiv 0$; also let $\alpha=m\delta$, $\gamma=n\delta$ be the equations of AC′, A′B′; and let P, Q, R be the points of intersection of the opposite sides BC, B′C′; CA′, C′A; A′B′, AB. Then Q is given by the equations $\alpha=m\delta$, $\gamma=0$; hence the equation of QB is $(m+1)\alpha+m\beta=0$, and B′ is given by this equation and $\gamma=n\delta$; therefore the equation of B′P is

$$(m+1)(\beta+\delta)-m\beta+n(m+1)\delta=0, \quad \text{or} \quad \beta+(m+1)(n+1)\delta=0.$$

But this being symmetrical with respect to m and n, and not involving α or γ, must, since we know that it is satisfied when $(m+1)\alpha+m\beta=0$ and $\gamma=n\delta$, be also satisfied when $(n+1)\gamma+n\beta=0$ and $\alpha=m\delta$; that is, PB′ passes through the intersection of RC and AQ, or R, C, C′ lie on a straight line.

II. Otherwise, let the diagonals meet, two and two, in L, M, N; then

$$\{Q\,.\,AMNA'\}=\{C'MLC\}=\{P\,.\,B'NLB\}; \quad \therefore \{AMNA'\}=\{BLNB'\},$$

and therefore AB, A′B′, CC′ meet in a point.

1527 (Proposed by the Rev. J. BLISSARD, B.A.)—If B be the representative of Bernouilli's numbers, so that $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, &c., and if $C_n f(x)$ denotes the coefficient of x^n in the expansion of $f(x)$; prove that

$$C_n\left\{3B_0\,.\,\frac{\log^2(1+x)}{1\,.\,2}+5B_2\,.\,\frac{\log^4(1+x)}{1\,.\,2\,.\,3\,.\,4}+7B_4\,.\,\frac{\log^6(1+x)}{1\,.\,2\,...\,6}+\&c.\right\}$$

$$=\frac{(-)^n}{n}\left\{\left(\frac{1}{1}+\frac{1}{2}+\dots\dots+\frac{1}{n}\right)^2+\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots\dots+\frac{1}{n^2}\right)-\frac{2}{n^2}\right\}.$$

Solution by the PROPOSER.

1. In the demonstration of this formula, it will be necessary for me to use the notation employed in my *Theory of Generic Equations* (see Vols. IV. and V. of the *Quarterly Journal of Pure and Applied Mathematics*). This notation, with the method grafted upon it, will be found, as I believe, to possess unrivalled power. The mode of its employment will be seen in what follows.

2. If $\dfrac{\theta}{e^\theta-1}$ (expanded in terms of θ) $=B_0+B_1\theta+B_2\dfrac{\theta^2}{1\,.\,2}+B_3\dfrac{\theta^3}{1\,.\,2\,.\,3}+\&c.$,

the quantities B_0, B_1, B_2, &c., are the celebrated numbers of Bernouilli. It is easy to see that $B_0=1$. The principle of what I have ventured to call (not unaptly, I hope) *Representative Notation* is to work (under due restrictions) with B^n for B_n. Hence, by Representative Notation, the above equation becomes

$$e^{B\theta}=\frac{\theta}{e^\theta-1}\;\dots\dots\dots\dots\dots\dots\dots\dots\dots\;(1).$$

It is to be observed, that if an equation subsists involving a Representative quantity as B, then in the final development B^n is to be replaced by B_n.

3. Multiply (1) by e^θ, then we have

$$e^{(B+1)\theta}=\frac{\theta e^\theta}{e^\theta-1}=\theta+\frac{\theta}{e^\theta-1}=\theta+e^{B\theta};\quad\therefore\; e^{(B+1)\theta}-e^{B\theta}=\theta.$$

Hence, expanding and equating coefficients, $(B+1)^n-B^n=0\;(n>1)\dots\dots(2)$. This is what I call the Generic Equation for the B-numbers, *i. e.*, the numbers of Bernoulli. Again, for θ put $-\theta$, then

$$\frac{-\theta}{e^{-\theta}-1}\left(=\frac{\theta e^\theta}{e^\theta-1}=e^{(B+1)\theta}\right)=1-B_1\theta+B_2\frac{\theta^2}{1\,.\,2}-B_3\frac{\theta^3}{1\,.\,2\,.\,3}+\&c.;$$

$$\therefore\; e^{(B+1)\theta}-e^{B\theta}\,(=\theta)=-2\left(B_1\theta+B_3\frac{\theta^3}{1\,.\,2\,.\,3}+\&c.\right);$$

$$\therefore\; B_1=-\frac{1}{2}\;\text{ and }\;B_n=0\;(n\text{ odd and }>1).$$

Hence also, from (2), $(B+1)^n=0$ (n odd and >1) $\dots\dots\dots\dots\dots\dots$ (3).

4. Having, as was necessary, premised thus much, assume P to be such a Representative quantity that $(1+x)^P=\dfrac{1}{1^2}-\dfrac{x}{2^2}+\dfrac{x^2}{3^2}-\&c.\;\dots\dots\dots(4).$

The values of the P quantities may be found, if required, by equating coefficients. Thus, $P_1 = -\frac{1}{4}$, $\frac{P(P-1)}{1.2} = \frac{1}{9}$;

$$\therefore\ P(P-1) = \frac{2}{9},\ \text{ that is, }\ P_2 - P_1 = \frac{2}{9},\ \ \therefore\ P_2 = -\frac{1}{36};$$

$$\frac{P(P-1)(P-2)}{1.2.3} = -\frac{1}{16},\ \ \therefore\ P_3 - 3P_2 + 2P_1 = -\frac{3}{8},\ \text{ and }\ P_3 = \frac{1}{24};$$

and so on. These values satisfy the equation (5) below.

A very important property belonging to Representative quantities is, that we may differentiate or integrate such an equation as (4) with respect to any actual quantity as x. Thus, integrating, we have

$$\frac{(1+x)^{P+1}-1}{P+1} = \frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c. = L^3(1+x),$$

with the usual notation of Spence's Transcendents. For x put $\frac{-x}{1+x}$, then

$$\frac{(1+x)^{-(P+1)}-1}{P+1} = -\left\{\frac{1}{1^3}\left(\frac{x}{1+x}\right) + \frac{1}{2^3}\left(\frac{x}{1+x}\right)^2 + \&c.\right\} = L^3\left(\frac{1}{1+x}\right);$$

$$\therefore\ L^3(1+x) + L^3\left(\frac{1}{1+x}\right) = \frac{1}{P+1}\left\{(1+x)^{P+1} + (1+x)^{-(P+1)} - 2\right\}$$

$$= 2\left\{\frac{P+1}{1.2}\log^2(1+x) + \frac{(P+1)^3}{1.2.3.4}\log^4(1+x) + \&c.\right\}\ \ \ldots\ldots\ldots\ (\text{I.})$$

We have now to determine $(P+1)^{2n+1}$. In (1) put $e^\theta = 1+x$, then

$$(1+x)^B = \frac{1}{x}\log(1+x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \&c.;$$

hence, integrating with respect to x,

$$\frac{(1+x)^{B+1}-1}{B+1} = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c. = x(1+x)^P.$$

Now for x put $e^\theta - 1$, then $(e^\theta - 1)e^{P\theta} = \frac{e^{(B+1)\theta}-1}{B+1}$;

$$\therefore\ e^{P\theta} = \frac{1}{(B+1)\theta}\left\{e^{(B+1)\theta}\frac{\theta}{e^\theta - 1} - \frac{\theta}{e^\theta - 1}\right\}$$

$$= \frac{1}{(B+1)\theta}\left\{e^{(B+1)\theta}.\,e^{B'\theta} - e^{B'\theta}\right\},$$

by putting $\frac{\theta}{e^\theta - 1} = e^{B'\theta}$, the B' being accented in order that it may have a separate development owing to its being connected with another function of B. Hence, multiplying by e^θ,

$$e^{(P+1)\theta} = \frac{1}{(B+1)\theta}\left\{e^{(B+1+B'+1)\theta} - e^{(B'+1)\theta}\right\}.$$

Expanding and equating coefficients,

$$(P+1)^n = \frac{1}{n+1}\left\{\frac{(B+1+B'+1)^{n+1}}{B+1} - \frac{(B'+1)^{n+1}}{B+1}\right\}$$

$$= \frac{1}{n+1}\left\{(B+1)^n + (n+1)(B+1)^{n-1}(B'+1) + \frac{(n+1)n}{1\,.\,2}(B+1)^{n-2}(B'+1)^2\right.$$

$$\left. + \ldots\ldots + \frac{(n+1)n}{1\,.\,2}(B+1)(B'+1)^{n-1} + (n+1)(B'+1)^n\right\},$$

which, when n is odd, is by (3) reduced to two terms, viz.,

$$(B+1)^{n-1}(B'+1) + \frac{n}{2}(B+1)(B'+1)^{n-1}.$$

But $B+1 = \frac{1}{2} = B'+1$, and $(B+1)^{n-1} = B_{n-1} = (B'+1)^{n-1}$ $(n>2)$,

$\therefore$ (n odd), $(P+1)^n = \left(\frac{1}{2}+\frac{n}{4}\right)B_{n-1}$, or, putting $2n+1$ for n,

$$(P+1)^{2n+1} = \frac{1}{4}(2n+3)\,B_{2n}. \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Hence, substituting in (I), we have

$$L^3(1+x) + L^3\left(\frac{1}{1+x}\right) = \frac{1}{2}\left\{\frac{3B_0}{1\,.\,2}\log^2(1+x) + \frac{5B_2}{1\,.\,2\,.\,3\,.\,4}\log^4(1+x)\right.$$

$$\left. + \frac{7B_4}{1\,.\,2\ldots.6}\log^6(1+x) + \&c.\right\} \ldots\ldots\ldots\ldots (II.)$$

5. Again, $C_n L^3(1+x) = (-)^{n+1}\,.\,\frac{1}{n^3}$, and

$$C_n L^3\left(\frac{1}{1+x}\right) = -C_n\left\{\frac{1}{1^3}\left(\frac{x}{1+x}\right) + \frac{1}{2^3}\left(\frac{x}{1+x}\right)^2 + \&c.\right\}$$

$$= \frac{(-)^n}{n}\left\{\frac{n}{1}\,.\,\frac{1}{1^2} - \frac{n(n-1)}{1\,.\,2}\,.\,\frac{1}{2^2} + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,.\,\frac{1}{3^2} - \&c.\right\}.$$

Now, in the *Quarterly Journal*, Vol. V., p. 328, it is shown that if

$\phi_1(n) = \frac{1}{1} + \frac{1}{2}\,\&c. + \frac{1}{n}$, $\phi_2(n) = \frac{\phi_1(1)}{1} + \frac{\phi_1(2)}{2}\,\&c.$; then

$$\phi_2(n) = \frac{n}{1}\,.\,\frac{1}{1^2} - \frac{n(n-1)}{1\,.\,2}\,.\,\frac{1}{2^2} + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,.\,\frac{1}{3^2} - \&c.,$$

$\therefore$ $C_n L^3\left(\frac{1}{1+x}\right) = \frac{(-)^n}{n}(\phi_2 n)$. Again, on p. 327, it is shown that

$\phi_2(n) = \frac{1}{2}\left\{(\phi_1 n)^2 + \psi_2 n\right\}$, where $\psi_2(n) = \frac{1}{1^2} + \frac{1}{2^2} + \ldots. + \frac{1}{n^2}$.

Hence $C_n\left\{L^3(1+x) + L^3\left(\frac{1}{1+x}\right)\right\} = \frac{(-)^n}{2n}\left\{(\phi_1 n)^2 + \psi_2 n - \frac{2}{n^2}\right\}$;

$$\therefore \text{ from (II.), } C_n\left\{\frac{3B_0}{1.2}\log^2(1+x)+\frac{5B_2}{1.2.3.4}\log^4(1+x)+\&c.\right\}$$

$$=\frac{(-)^n}{n}\left\{\left(\frac{1}{1}+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right)^2+\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots\ldots+\frac{1}{n^2}\right)-\frac{2}{n^2}\right\}.$$

1539 (Proposed by the Rev. T. P. Kirkman, A.M., F.R.S.)—To form all the groups of six powers, $1, \theta, \theta^2, \ldots \theta^5$, which have the common substitution of the second order $\theta^3 = 361542$.

Solution by the Proposer.

Every group of n powers, when n is a composite number, can be formed by involution of the powers of θ, any substitution of the nth order in the group; or by evolution of the roots of θ^e, any substitute of an inferior order in the group. We have here $\theta^e = \theta^3 = 361542$, which has three circular factors

$$p_1 = 13, \quad p_2 = 26, \quad p_3 = 45,$$

and which completes with *unity* the group

$$\begin{array}{c} 123456 \\ 361542 \end{array} \qquad G.$$

The process of evolution of the cube roots of 361542 is exactly that of completing G into a grouped group of the 6th order. We require an auxiliary or operative group having as many elements as G has circular factors, that is, three. The only operative group that we can use is

$$\begin{array}{c} 123 \\ 231 \\ 312 \end{array} \qquad (B).$$

We form the two substitutions

$$\frac{p_2p_3p_1}{p_1p_2p_3} = \frac{264513}{132645} = 246135 = \gamma,$$

$$\frac{p_3p_1p_2}{p_1p_2p_3} = \frac{451326}{132645} = 415263 = \delta;$$

and our grouped group is G and two derivates of G, which are two derived derangements of G,

$$G + \gamma G + \delta G = \begin{array}{c} 123456 \\ 361542 \end{array} \quad \begin{array}{c} 246135 \\ 652314 \end{array} \quad \begin{array}{c} 415263 \\ 534621 \end{array} \qquad H.$$

This, constructed by involution on $\theta = 652314$, is

$$\left.\begin{array}{l} 123456 \\ 652314 = \theta \\ 415263 = \theta^2 \\ 361542 = \theta^3 \\ 246135 = \theta^4 \\ 534621 = \theta^5 \end{array}\right. \qquad H,$$

where the student will observe that involution is merely the completion under *unity* of a given vertical circular factor. This is one of the groups of six powers which have $\theta^3 = 361452$. To find the rest we must modify (B), and give it the forms following:

$$\begin{array}{lll} 123 & 123 & 123 \\ 2^2 31 & 23^2 1 & 231^2 \\ 3^2 12^2 & 3^2 1^2 2 & 31^2 2^2 \\ (B_1) & (B_2) & (B_3) \end{array}$$

These are three *adfected operative groups.*

DEF. *An adfected substitution* shows one or more exponents > 1 over some of its elements. Adfected substitutions have no use or meaning, except as operatives, employed in the formation of grouped groups.

OBS. A. The exponents in an adfected auxiliary are estimated according to a modulus r, which is the order (= number of elements) of the circular factors in the subject of operation; so that, m being any adfected element, $m^{r+i} = m^i$. In our problem $r = 2$.

OBS. B. To form the product $C = AB$ of two adfected substitutions, the rule is: Pronounce the sinister factor A, and write it down with its exponents exactly in the order in which you see 1234 . . . arranged in the dexter B, taking care to increase by e the exponent of the kth element of A, whenever k in B carries the exponent $e + 1$.

OBS. C. If all the exponents of an adfected substitution ϵ be increased by the same number, ϵ is not altered in value, *i. e.*, the result of operation with ϵ on the subject is unchanged, as we shall presently understand.

The student who has read my *Hints on the Theory of Groups* (*Messenger of Mathematics*, Vol. i., pp. 58—68) will easily satisfy himself by considering the Obs. A, B, C, that (B_1), (B_2), (B_3) are all groups; *e.g.*,

$$2^2 31\,(B_1) = 2^2 31,\ 3^2 12^2,\ 1^2 2^2 3^2, = (B_1).$$

We have now only to define that, if $p_m = abcd \ldots$ is any circula rfactor, $p_m^2 = bcd \ldots a$, $p_m^3 = cd \ldots ab$, &c., and we can proceed to form our grouped groups on the three adfected auxiliaries.

By (B_1) we construct the derivants (= sinister multipliers of G),

$$\frac{p_2^2 p_3 p_1}{p_1 p_2 p_3} = \frac{624513}{132645} = 642135 = \gamma_1,$$

$$\frac{p_3^2 p_1 p_2^2}{p_1 p_2 p_3} = \frac{541362}{132645} = 514623 = \delta_1;$$

which give the grouped group

$$\begin{array}{rlcll} G + \gamma_1 G + \delta_1 G = & 123456 & \text{or} & 123456 & \\ & 361542 & & 642135 = \theta & \\ & 642135 & & 514623 = \theta^2 & \\ & 256314 & & 861542 = \theta^3 & H_1. \\ & 514623 & & 256314 = \theta^4 & \\ & 425261 & & 435261 = \theta^5 & \end{array}$$

By (B_2) we form the derivants

$$\frac{p_2 p_3^2 p_1}{p_1 p_2 p_3} = \frac{265413}{132645} = 256134 = \gamma_2,$$

$$\frac{p_3^2 p_1^2 p_2}{p_1 p_2 p_3} = \frac{543126}{132645} = 534261 = \delta_2,$$

which give the grouped group

$$G+\gamma_2 G+\delta_2 G = \begin{matrix}123456\\361542\\256134\\642315\\534261\\415623\end{matrix} \quad\text{or}\quad \begin{matrix}123456 & \\ 256134 & =\theta \\ 534261 & =\theta^2 \\ 361542 & =\theta^3 \\ 642315 & =\theta^4 \\ 415623 & =\theta^5\end{matrix} \qquad H_2.$$

By (B_3) we obtain

$$\frac{p_2p_3p_1^3}{p_1p_2p_3} = \frac{264531}{132645} = 246315 = \gamma_3,$$

$$\frac{p_3p_1^2p_2^3}{p_1p_2p_3} = \frac{453162}{132645} = 435621 = \delta_3,$$

giving the grouped group

$$G+\gamma_3 G+\delta_3 G = \begin{matrix}123456\\361542\\246315\\652134\\435621\\514263\end{matrix} \quad\text{or}\quad \begin{matrix}123456 & \\ 246315 & =\theta \\ 435621 & =\theta^2 \\ 361542 & =\theta^3 \\ 652134 & =\theta^4 \\ 514263 & =\theta^5\end{matrix} \qquad H_3.$$

H, H_1, H_2, H_3 are the groups demanded by the Question, and there are no more. If we take for θ^3 in turn each of the 15 substitutions that have three transpositions, we shall by the same direct process construct 60 groups of six powers, which are all such groups possible; for there are but 120 ways of writing 23456 under 1, and each group H, being the group of powers either of θ or θ^5, is produced in two of these ways. That is, there are just 60 groups H. The general investigation of the number of equivalent groups of k powers made with n elements may be seen in § 3 of my "Memoir on the Theory of Groups and Many-valued Functions," in the *Manchester Memoirs*, Ser. iii., Vol. 1. The subject of grouped groups is handled at length in the 7th and 8th Sections.

If in (B_3) we chose to write $3^2 12$ instead of $31^2 2^2$, we should have obtained for δ_3

$$\frac{p_3^2p_1p_2}{p_1p_2p_3} = \frac{541326}{132645} = 514263,$$

which would have made no difference in $\delta_3 G$ or H_3. Thus we see the truth of Obs. C.

All the 60 groups H, H_1, &c. can be constructed also by evolution of the square roots of a substitution like $\theta^2 = 435621$ of the third order.

We have found eight cube roots θ of the given θ^3. If in any algebraic function of six elements $x_1, x_2, \ldots x_6$ we perform three times the substitution θ on the sub-indices, the result is identical with that obtained by the substitution $361542 = \theta^3$, once effected.

1502 (Proposed by Professor SYLVESTER.)—If $\tan\theta = \dfrac{\sin\alpha\cos\lambda - \sin\beta\sin\lambda}{\cos\alpha\cos\lambda - \cos\beta\sin\lambda}$, and $\tan\phi = \dfrac{\sin\alpha\sin\lambda - \sin\beta\cos\lambda}{\cos\alpha\sin\lambda - \cos\beta\cos\lambda}$, show that $\theta+\phi$ is independent of λ.

Solution by Mr. S. Bills.

Put $\tan\theta = \dfrac{m}{n}$, $\tan\phi = \dfrac{p}{q}$, then $\tan(\theta+\phi) = \dfrac{mq+np}{nq-mp}$ ....(1).

Now, if m, n, p, q have the values in the question, we shall have

$$mq + np = \sin\lambda\cos\lambda\,(\sin 2\alpha + \sin 2\beta) - \sin(\alpha+\beta)$$

$$= \sin(\alpha+\beta)\left\{\sin 2\lambda\cos(\alpha-\beta) - 1\right\} \text{ ....(2).}$$

In like manner we find

$$nq - mp = \cos(\alpha+\beta)\left\{\sin 2\lambda\cos(\alpha-\beta) - 1\right\} \text{ ....(3).}$$

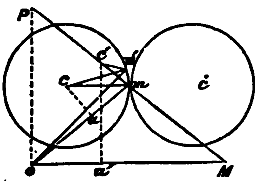
Substituting from (2) and (3) in (1), we have

$$\tan(\theta+\phi) = \tan(\alpha+\beta), \text{ or } \theta+\phi = \alpha+\beta,$$

which is independent of λ.

1464 (Proposed by Dr. Hirst, F.R.S.)—If ϵ denote the ratio of the segments into which any radius vector of a curve is divided by projecting the centre of curvature upon it orthogonally, prove that, by a similar operation, the corresponding radius vector of the nth pedal will be divided into segments whose ratio is $n+\epsilon$.

Solution by the Proposer.



Let o be the origin, (c) a fixed circle, and (c) an equal circle rolling around (c). Connect any point of contact m with the origin o and with the centre c; make $c\hat{m}P = c\hat{m}o$, and produce Pm to M, so that mM $= mo$: then Mm will be the normal at M to the epitrochoid described by the point M, which latter, with respect to the rolling circle, is the homologue of o, relative to the fixed circle. Again, if the adjacent point of contact m' of the two circles be connected with o, and with c, and $c\hat{m}'c'$ be made equal to $c\hat{m}'o$, $m'c'$ will be an adjacent normal of the epitrochoid, and c' its centre of curvature. Now the angle $m\hat{c}'m'$ is obviously equal to $2m\hat{c}m' - m\hat{o}m'$, or more simply $\hat{c}' = 2\hat{c} - \hat{o}$; and since $c'm'$ and om' are equally inclined to the element mm', it is readily seen that

$$c'm \,.\, \hat{c}' = om \,.\, \hat{o} = cm \,.\, \hat{c} \,.\, \cos cmo;$$

or, if a be the orthogonal projection of c on the radius vector om,

$$\frac{c'm}{om} = \frac{\hat{o}}{2\hat{c}-\hat{o}}, \text{ and } \frac{am}{om} = \frac{\hat{o}}{\hat{c}}; \quad \therefore \frac{om + c'm}{om - c'm} = \frac{\hat{c}}{\hat{c}-\hat{o}} = \frac{om}{oa}.$$

But if oP and $c'a'$ be drawn perpendicular to oM, the first of the above ratios, since $mP = mo - mM$, is clearly equal to $c'M : Pc'$ or $a'M : oa'$; so that the equation may be thus written:

$$\frac{a'M}{oa'} = \frac{am}{oa} + 1.$$

With respect to the epitrochoid and the circle, these two ratios are precisely those referred to in the theorem. But, as is well known, the epitrochoid is *similar* to the first positive pedal, so that our theorem is true for a primitive circle and its pedal. Further, if we consider this circle as the osculating circle of any primitive curve, its pedal will clearly have three-pointic contact with the pedal of that curve, and the above relation, which may now be written $\epsilon_1 = \epsilon + 1$, will hold for any curve whatever and its first positive pedal; so that, by successive deductions, we have finally the formula $\epsilon_n = \epsilon + n$; which was to be demonstrated, and which is manifestly true for all negative as well as positive integral values of n.

One or two consequences of this theorem deserve notice, *First.* If ϵ be constant for every point of the primitive curve, every pedal of the latter will possess the same characteristic property. The polar equation of such a primitive has always the form $r = a \cos^{\epsilon} \frac{\theta}{\epsilon}$, hence that of its nth pedal will be $r_n = a \cos^{\epsilon+n} \frac{\theta}{\epsilon+n}$. *Secondly,* The logarithmic spiral is *characterized* by the relation $\epsilon = \infty$; whence we conclude, not only that every pedal of a logarithmic spiral is itself a spiral of the same nature, but also that the *ultimate pedal* ($n = \pm\infty$) *of any primitive curve whatever is a logarithmic spiral.*

N.B.—Shortly after publishing this Theorem in *Tortolini's Annali* (1859), I found that Prof. Maxwell had previously established the same, though in a different manner, in the Transactions of the Royal Society of Edinburgh.

1541 (Proposed by H. J. PURKISS, B.A.) — Rays of light diverge from a point S, and are reflected at any given plane curve. Show that if P be any point on the curve, and Q the corresponding point of the caustic,

$$SQ = \frac{2r\,(r^2 - 2r\rho \sin^3\phi + \rho^2 \sin^4\phi)^{\frac{1}{2}}}{2r - \rho \sin\phi}, \quad SQ \sin QSP = \frac{2r\rho \sin^2\phi \cos\phi}{2r - \rho \sin\phi};$$

where $SP = r$, $\rho =$ the radius of curvature at P, and $\phi =$ the angle between SP and the tangent at P.

Solution by the PROPOSER; *and* F. D. THOMSON, M.A.

We know that $\frac{1}{PQ} + \frac{1}{r} = \frac{2}{\rho \sin\phi}$, $\therefore$ $PQ = \frac{r\rho \sin\phi}{2r - \rho \sin\phi}$;

also $\angle SPQ = \pi - 2\phi$, $\therefore$ $SQ \sin QSP = PQ \sin SPQ = \dfrac{2r\rho' \sin^2\phi \cos\phi}{2r - \rho\sin\phi}$.

$$\text{Again, } SQ^2 = SP^2 + PQ^2 + 2SP \,.\, PQ \cos 2\phi$$

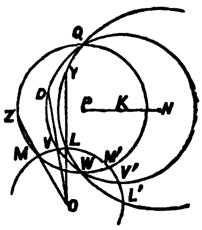
$$= r^2 \left\{ 1 + \frac{\rho^2 \sin^2\phi}{(2r-\rho\sin\phi)^2} + \frac{2\rho\sin\phi\cos 2\phi}{2r-\rho\sin\phi} \right\},$$

$$= \frac{2r\,(r^2 - 2r\rho\sin^3\phi + \rho^2\sin^4\phi)^{\frac{1}{2}}}{2r-\rho\sin\phi}.$$

[NOTE.—The formula at the beginning of the Solution is one of two which were first given by the Marquis de l'Hôpital (*Analyse des Inf. Petits*, 1696). An investigation of the formula may be seen on p. 78 of Serret's *Méthodes en Géom.*, where however $\frac{1}{\rho}$ is incorrectly put instead of $\frac{2}{\rho}$.—ED.]

1540 (Proposed by Mr. A. RENSHAW.)—Two circles, whose centres are P and N, cut one another in Q and W, and a third circle is described with radius KQ or KW, K being the middle point of PN; also from *any* point O as centre a fourth circle is drawn cutting the aforesaid three circles, viz. (P) in M and M′, (N) in L and L′, (K) in V and V′; and OM, OL, OV are joined and produced to meet the circles (P), (N), (K) again in Z, Y, D respectively. Prove that LY + MZ = 2VD.

Solution by MR. D. M. ANDERSON; F. D. THOMSON, M.A.; *and the* PROPOSER.



Suppose OP, OK, ON to meet the circles (P), (K), (N), respectively, in A, A′; B, B′; C, C′; then, since K is the middle of PN,

$$2KN^2 = PQ^2 + NQ^2 - 2KQ^2 = PO^2 + NO^2 - 2KO^2;$$

$$\therefore\ (PO^2 - PA^2) + (NO^2 - NC^2) = 2\,(KO^2 - KB^2),$$

$$\text{or } OA \,.\, OA' + OC \,.\, OC' = 2OB \,.\, OB',$$

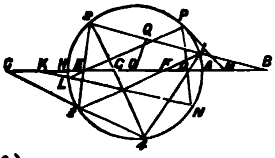
$$\text{or } OM \,.\, OZ + OL \,.\, OY = 2OV \,.\, OD.$$

$$\text{But } OM = OL = OV;$$

$$\therefore\ OZ + OY = 2OD, \text{ and } LY + MZ = 2VD.$$

1549 (Proposed by CANTAB.)—If through any two points on the diameter of a circle, equidistant from the centre, any two chords be drawn, the straight lines passing through their extremities will intersect the diameter (produced, if necessary) in points also equidistant from the centre.

Solutions (I.) *by the* REV. H. HOLDITCH; (II.) *by* F. D. THOMSON, M.A.; MR. D. M. ANDERSON; *and* MR. A. RENSHAW.



I. Let 1234 be any quadrilateral inscribed in a circle whose centre is O and radius unity; and let the lines 12, 34; 14, 23; 13, 24 meet the diameter AOH in B, C; D, E; F, G. Draw OQ perpendicular to 12, and let $\angle$ AO1 $= \theta_1$, $\tan \frac{1}{2}\theta_1 = t_1$, &c.; then

$$OQ = \cos QO1 = \cos \tfrac{1}{2}(\theta_2 - \theta_1),$$

$$OB = OQ \sec QOB = \cos \tfrac{1}{2}(\theta_2 - \theta_1) \sec \tfrac{1}{2}(\theta_2 + \theta_1),$$

or $OB = \dfrac{1+t_1t_2}{1-t_1t_2}$; similarly $OC = \dfrac{1+t_3t_4}{1-t_3t_4}$; $\therefore\ OB + OC = \dfrac{2(1-t_1t_2t_3t_4)}{(1-t_1t_2)(1-t_3t_4)}$;

similarly $OD + OE = \dfrac{2(1-t_1t_2t_3t_4)}{(1-t_1t_4)(1-t_2t_3)}$, and $OF + OG = \dfrac{2(1-t_1t_2t_3t_4)}{(1-t_1t_3)(1-t_2t_4)}$.

If therefore any two of the three quantities, OB + OC, OD + OE, OF + OG, is zero, we must have $1 - t_1t_2t_3t_4 = 0$, and the other two will be also zero.

COR. Hence a tangent at any point P may be drawn by taking any two points D and E in a diameter equidistant from the centre, then drawing PDN, PEL, NLK (L, N being on the circle), making OM = OK, and joining PM.

II. Otherwise: suppose OF = OG, and therefore HF = AG; then

$$\{1.A3H2\} = \{4.A3H2\};$$

$$\therefore\ \{AFHB\} = \{ACHG\},\ \text{ or }\ \frac{AH}{HF} : \frac{AB}{BF} = \frac{AH}{HC} : \frac{AG}{GC};$$

$$\therefore\ FB : BA = GC : CH,\ \text{ and }\ FB : FA = GC : GH;$$

but FA = GH, therefore FB = GC and OB = OC.

1559 (Proposed by the EDITOR.)—Eliminate x and y from the equations

$$\frac{x(x^2+y^2-3a^2)+2a^3}{3\xi\{(x-a)^2+y^2\}} = \frac{y(x^2+y^2-3a^2)}{3v\{(x-a)^2+y^2\}} = \frac{(x^2+y^2-a^2)^2}{4\{(x-a)^2+y^2\}} = a^2.$$

Solution (I.) *by* W. SPOTTISWOODE, F.R.S.; (II.) *by* MR. S. BILLS.

I. Let $x^2+y^2-a^2=r^2$, then the given equations become

$$x(r^2-2a^2)+2a^3 = 3a^2\xi(r^2+2a^2-2ax) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$y(r^2-2a^2) = 3a^2v(r^2+2a^2-2ax) \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$r^4 = 4a^2(r^2+2a^2-2ax) \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

whence $(r^2-2a^2)\,x = \frac{3}{4}r^4\xi-2a^3 \ldots\ldots (4)$; $(r^2-2a^2)\,y = \frac{3}{4}r^4 v \ldots\ldots (5)$;

$\therefore\ (r^2-2a^2)^2(x^2+y^2) = (r^2-2a^2)^2(r^2+a^2) = \frac{9}{16}r^8(\xi^2+v^2)-3a^3r^4\xi+4a^6 \ldots (6).$

Also, from (3) and (4), $x = \dfrac{3r^4\xi-8a^3}{4\,(r^2-2a^2)} = \dfrac{8a^4+4a^2r^2-r^4}{8a^3} \ldots\ldots\ldots (7).$

But multiplying out, and dividing throughout by r^4, (6) and (7) become

$9\,(\xi^2+v^2)\,r^4-16r^2+48a^2(1-a\xi) = 0 \ldots (8)$; $r^2-6a^2(1-a\xi) = 0 \ldots (9)$;

$$\text{also } \big\{(8)+8\,(9)\big\} \div r^2 \text{ gives } 9\,(\xi^2+v^2)\,r^2-8 = 0,$$

whence finally, $27a^2(1-a\xi)(\xi^2+v^2)-4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (C).$

II. Otherwise: assume $x^2+y^2-a^2=2at$, and $(x-a)^2+y^2=2au$; then

$$x=a+t-u,\quad y^2=2au-(t-u)^2,\quad x^2+y^2-3a^2=2a\,(t-a);$$

hence, by substitution, the given equations become

$$\frac{t^2-u\,(t-a)}{3\xi u} = a^2 \ldots (1);\quad \frac{\big\{2au-(t-u)^2\big\}(t-a)^2}{9v^2u^2} = a^4 \ldots (2);\quad \frac{t^2}{2au}=1 \ldots (3).$$

From (1) and (3) we find $t = 3a\,(1-a\xi)$, $u = \frac{9}{2}a\,(1-a\xi)^2$;

and substituting these values in (2), we obtain

$$27a^2(1-a\xi)(\xi^2+v^2) = 4 \ldots\ldots\ldots\ldots (C).$$

[NOTE.—The foregoing elimination furnishes us with a direct investigation (by the aid of Dr. Booth's formulæ) of the *tangential* equation of the *Cardioid*, otherwise deduced in the *Note* to the Solution of Quest. 1509. (See *Educational Times* for August.) For the *projective* equation being (see Salmon's *Higher Curves*, Art. 226)

$$V \equiv (x^2+y^2-a^2)^2-4a^2\big\{(x-a)^2+y^2\big\} \equiv (x^2+y^2-3a^2)^2+4a^3(2x+3a) = 0,$$

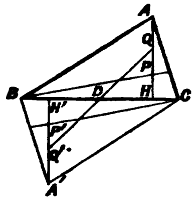
$$\text{we have } \frac{dV}{dx} = 4x\,(x^2+y^2-3a^2)^2+8a^3,\quad \frac{dV}{dy} = 4y\,(x^2+y^2-3a^2);$$

$$\therefore\ \frac{dV}{dx}x+\frac{dV}{dy}y = 12a^2\big\{(x-a)^2-y^2\big\};$$

hence the equations in the Question are those from which, by eliminating x and y, we obtain the *tangential* equation (C) of the *Cardioid*.—EDITOR.]

1552. (Proposed by J. GRIFFITHS, M.A.)—Through each angular point of any triangle ABC let a line be drawn parallel to the opposite side; and let A′, B′, C′ be the vertices opposite to A, B, C, of the new triangle thus formed: it is required to prove that the nine-point circle of the triangle ABC touches the nine-point circles of the triangles A′BC, B′CA, C′AB at the middle points of the sides BC, CA, AB respectively.

***Solution by* F. D. THOMSON, M.A.; *and* MR. H. MURPHY.**



Draw AH, A′H′ perpendicular to BC; and let P, P′ be the points of intersection of the perpendiculars of the two triangles ABC, A′BC, and D, Q, Q′ the middle points of BC, AP, A′P′. Join QD, Q′D; then QD is a diameter of the nine-point circle of ABC (McDowell's *Exercises*, p. 56), and Q′D of that of A′BC: hence the theorem will be proved if QDQ′ is shown to be a straight line. Now from the symmetry of the figure it is evident that QHD, Q′H′D are similar triangles, hence QD and Q′D are in a straight line, and therefore the two circles have the same tangent at D, viz., the perpendicular to the straight line QDQ′.

1561 (Proposed by PROFESSOR SYLVESTER.)—Give a barycentric proof of the condition for four points a, b, c, d lying in a circle; viz.,

$$bcd\,.\,aP^2 - cda\,.\,bP^2 + dab\,.\,cP^2 - abc\,.\,dP^2 = 0.$$

Solution by the PROPOSER.

Let $abcd$ be any quadrilateral whatever, and let bcd, $-cda$, dab, $-abc$ be called A, B, C, D respectively; also let A, B, C, D be parallel forces supposed to act at a, b, c, d. We have $(A+C) = -(B+D)$; and it will be at once obvious, on drawing the diagonals of the quadrilateral, that the centre of gravity of A, C, and also of B, D, will lie at the intersection of the same. Hence the statical effect of the four forces A, B, C, D will be absolutely *nil*.

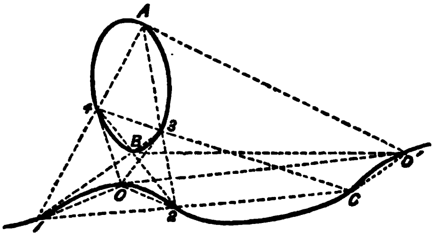
Let now a, b, c, d be the points supposed in the Question, P any fifth point taken arbitrarily, and O the centre of the circle. Then, taking moments about a line through O perpendicular to OP, we obtain, by virtue of the preceding remark, $\Sigma(A\,.\,Oa\,.\,\cos aOP) = 0$; hence

$$\Sigma A\,(Oa^2 + OP^2 - 2Oa\,.\,OP\cos aOP) = 0,$$

that is, $\Sigma(A\,.\,aP^2) = 0$, as was to be proved.

1562 (Proposed by F. D. THOMSON, M.A.)—Find the locus of the points of contact of tangents drawn from a given point to a conic circumscribing a given quadrangle. The quadrangle being supposed convex, trace the changes of form of the locus for different positions of the given point.

Solution by PROFESSOR CAYLEY; *and the* PROPOSER.



Let O be the given point; 1, 2, 3, 4 the vertices of the given quadrangle; A, B, C the centres of the quadrangle, viz., A the intersection of the lines 14, 23; B of 24, 31; C of 34, 12. The polars of O in regard to the several circumscribed conics intersect in a point O′. This being so, the locus is a cubic passing through the nine points 1, 2, 3, 4, A, B, C, O, O′, and which is moreover such that the tangents at the four points 1, 2, 3, 4 meet the cubic in the point O, and the tangents at the four points A, B, C, O meet the cubic in the point O′. It is to be remarked that the nine points are so related to each other that a cubic through any eight of these points passes through the remaining ninth point; say a cubic through 1, 2, 3, 4, A, B, C, O passes through O′; the nine points consequently do not determine the cubic; but the cubic will be determined, *e.g.*, by the conditions that it passes through 1, 2, 3, 4, A, B, C, O, and has O1 for the tangent at 1. The series of cubics corresponding to different positions of the point O is identical with the series of cubics passing through the seven points 1, 2, 3, 4, A, B, C. Conversely any given cubic curve may be taken to be a cubic of the series; and the points 1, 2, 3, 4 will then be determined as follows, viz., 1, 2, 3, 4 are the points of contact of the tangents to the cubic from an arbitrary point O on the cubic; and then taking as before A, B, C for the intersections of 14, 23, of 24, 31 and of 34, 12, respectively, the points A, B, C will lie on the cubic, and the tangents at A, B, C, O will meet the cubic in a point O′. I call to mind that a cubic curve without singularities is either *complex* or simplex; in the simplex kind there can be drawn from any point of the curve two, and only two, real tangents to the curve; in the complex kind, there can be drawn four real tangents or else no real tangent, viz. from any point on a certain branch of the curve there can be drawn four real tangents, from a point on the remaining portion of the curve no real tangent. Hence, in the foregoing construction, in order that the points 1, 2, 3, 4 may be real, the given cubic must be of the complex kind, and the point O must be taken on the branch which has through each of its points four real tangents.

The foregoing results may be established *geometrically* or *analytically*; but for brevity I merely indicate the analytical demonstration. Suppose first, that the points 1, 2, 3, 4 are given as the intersections of the conics $U=0$, $V=0$; let α, β, γ be the coordinates of the point O, and write $D=\alpha\delta_x+\beta\delta_y+\gamma\delta_z$, so that $DU=0$ and $DV=0$ are the equations of the polars of O in regard to the conics $U=0$, $V=0$ respectively. The equation of any conic through the four points is $U+kV=0$; and the equation of the polar of O in regard thereto is $DU+kDV=0$; eliminating k from these equations, we have $UDV-VDU=0$, which is the equation of the given locus. We see at once that it is a cubic curve passing through the points $(U=0, V=0)$, that is, the points 1, 2, 3, 4; and through the point $DU=0$, $DV=0$, that is, the point O′; it also follows without difficulty that the curve passes through the point O. But for the remaining results it is better to particularize the conics $U=0$, $V=0$. Let the equations of 12, 23, 34, 41

be $x=0$, $y=0$. $z=0$, $w=0$ respectively, (where $x+y+z+w=0$); and in the same system, let $\alpha, \beta, \gamma, \delta$ be the coordinates of O ($\alpha+\beta+\gamma+\delta=0$), then $xz=0$, $yw=0$ are each of them a conic (pair of lines) passing through the four points; and we may therefore write $U=yw$, $V=xz$; the equation $UDV-VDU=0$ thus becomes $yw(\alpha z+\gamma x)-xz(\beta w+\delta y)=0$, or, as this equation may also be written,

$$\frac{\alpha}{x}-\frac{\beta}{y}+\frac{\gamma}{z}-\frac{\delta}{w}=0,$$

which is the equation of the cubic curve; and from this form the several above mentioned results may be obtained without difficulty.

To give an idea of the form of the curve corresponding to a given convex quadrangle 1234, and given position of the point O, I suppose that O is situate *within* the quadrangle, for instance in the triangle B12. The mere inspection of the figure, and consideration of the conditions which are to be satisfied by the cubic curve, is enough to show that this is of the form described by NEWTON as *anguinea cum ovali*, viz., the oval passes through the points 3, 4, A, B, and the serpentine branch through the points 1, 2, C, O, O′. But the complete discussion of the different cases would be somewhat laborious.

[A *geometrical* investigation of the locus is given on p. 124 of CREMONA'S *Teoria Geometrica delle Curve Piane.*—EDITOR.]

1564 (Proposed by the Rev. R. H. WRIGHT, M.A.)—Find the trilinear equations of the circles described on the sides of a triangle whose vertices are (i.) the feet of the perpendiculars from the angles of the triangle of reference on the opposite sides, (ii.) the middles of the sides, (iii.) the points in which the internal bisectors of the angles meet the opposite sides.

Solution by the EDITOR.

Let P, Q be the points in which the straight line $l\alpha+m\beta+n\gamma=0$ meets the sides AC, AB of the triangle of reference ABC; then the following system of equations will represent two straight lines PR, QR which pass through P, Q and meet at right angles in R; viz.,

$$l\alpha+\mu\beta+n\gamma=0, \qquad l\alpha+m\beta+\nu\gamma=0,$$

$$l^2+m\mu+n\nu-(mn+\mu\nu)\cos A-l(n+\nu)\cos B-l(m+\mu)\cos C=0;$$

the last being the condition of perpendicularity.

If we eliminate μ, ν between these three equations, the resulting equation will be that of the locus of R, that is to say, of *the circle on* PQ *as diameter.* The elimination is effected at once by substituting in the third equation the

values of μ, ν, derived from the first and second; and the resulting equation, viz.,

$$(l^2 \cos A)\,\alpha^2 + m\,(n - l \cos B)\,\beta^2 + n\,(m - l \cos C)\,\gamma^2$$
$$+ (2mn \cos A + nl \cos B + lm \cos C - l^2)\,\beta\gamma + l\,(m + n \cos A - l \cos C)\,\gamma\alpha$$
$$+ l\,(n + m \cos A - l \cos B)\,\alpha\beta = 0 \ldots\ldots\ldots\ldots\ldots (\text{I.}),$$

is that of the circle described on the part of the line (l, m, n) intercepted between the sides β and γ of the triangle of reference.

2. Equation (I.) may also be deduced as a particular case of the following *theorem*, which was given by Mr. Clifford, in his Solution of Quest. 1514.

If $A=0$, $B=0$, $C=0$, $D=0$ are the equations of any four lines in a plane, and if $\Psi AB=0$ denote the condition that A and B may be at right angles to each other, the equation of the circle whose diameter is the line joining the points (AB), (CD) is

$$\begin{vmatrix} \Psi AC, & \Psi AD, & A \\ \Psi BC, & \Psi BD, & B \\ C, & D, & 0 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots (\text{II.})$$

The proof of the theorem depends on the fact that the function ΨAB (whose evanescence is the condition that A and B may be at right angles to each other) is of the first degree in the coefficients of each of the lines A, B. If then we seek the locus of the intersection of $A = kB$, $C = k'D$, which are perpendicular to each other, we find in the first place

$$\Psi (A - kB,\ C - k'D) \equiv \Psi AC - k\Psi BC - k'\Psi AD + kk'\Psi BD = 0;$$

and then, substituting for k, k' their values (viz., $A : B$, $C : D$), the equation of the circle described on the line joining (AB), (CD), is found to be

$$BD\ \Psi AC - AD\ \Psi BC - BC\ \Psi AD + AC\ \Psi BD = 0 \ldots\ldots\ldots\ldots (\text{III.}),$$

which is the same thing as (II.)

Taking $A \equiv \beta$, $B \equiv l\alpha + n\gamma$, $C \equiv \gamma$, $D \equiv l\alpha + m\beta$; we have $\Psi AC = -\cos A$, $\Psi AD = m - l \cos C$, $\Psi BC = n - l \cos B$, $\Psi BD = l^2 - mn \cos A - nl \cos B - lm \cos C$; and with these values equation (II.) or (III.) becomes

$$(l\alpha + m\beta)(l\alpha + n\gamma) \cos A + (n - l \cos B)(l\alpha + m\beta)\,\beta + (m - l \cos C)(l\alpha + n\gamma)\,\gamma$$
$$+ (mn \cos A + nl \cos B + lm \cos C - l^2)\,\beta\gamma = 0,$$

which is the same as (I.)

3. By aid of (I.) the equations required in the Question may be at once written down. Thus for (i.) we have $l : m : n = -\cos A : \cos B : \cos C$; hence the equation of the circle described on the side opposite to $(\beta\gamma)$ is

$$\alpha^2 \cos^3 A + \beta^2 \sin A \sin B \cos B + \gamma^2 \sin A \sin C \cos C$$
$$- \left\{\beta\gamma \cos A + \gamma\alpha \cos (A - C) + \alpha\beta \cos (A - B)\right\} \cos A = 0.$$

For (ii.) we have $l : m : n = -\sin A : \sin B : \sin C$; hence the equation is

$$\alpha^2 \sin A \cos A + \beta^2 \sin B\,(\sin A \cos B + \sin C) + \gamma^2 \sin C\,(\sin A \cos C + \sin B)$$
$$- 2\,(\sin^2 A - \cos A \sin B \sin C)\,\beta\gamma - 2\,(\sin A \sin B)\,\gamma\alpha - 2\,(\sin A \sin C)\,\alpha\beta = 0.$$

For (iii.) we have $l:m:n=-1:1:1$; hence the equation is

$$\alpha^2\cos A+\beta^2(1+\cos B)+\gamma^2(1+\cos C)-(1-2\cos A+\cos B+\cos C)\,\beta\gamma$$
$$-(1+\cos A+\cos C)\,\gamma\alpha-(1+\cos A+\cos B)\,\alpha\beta=0.$$

The equations of the circles on the sides respectively opposite to $(\gamma\alpha)$, $(\alpha\beta)$ may be deduced from these by an obvious interchange of the letters.

4. If in (I.) we put $n=0$, $l:m=\cos A:-\cos B$, or $\sin A:-\sin B$, or $1:-1$, we have

$$\alpha^2\cos^2 A+\beta^2\cos^2 B-\beta\gamma\sin B\sin C-\gamma\alpha\sin C\sin A-2\alpha\beta\cos A\cos B=0,$$

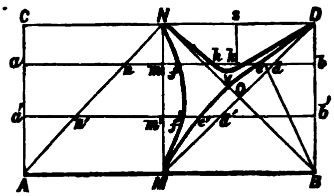
$$\alpha^2\sin A\cos A+\beta^2\sin B\cos B-\beta\gamma(\sin A+\sin B\cos C)$$
$$-\gamma\alpha(\sin B+\sin A\cos C)-\alpha\beta\sin C=0,$$

$$\alpha^2\cos A+\beta^2\cos B-(\beta\gamma+\gamma\alpha)(1+\cos C)-\alpha\beta(\cos A+\cos B)=0,$$

which are the respective equations of the circles whose diameters are the lines drawn from C perpendicular to AB, bisecting AB, and bisecting the angle C and terminated by AB.

1100 (Proposed by EXHUMATUS.)—A straight line is divided at random into three segments; find the probability that it may be possible to form, with these segments, an *acute-angled* triangle.

Solution by the EDITOR.



Let AB be the given line, M its middle point. On AM, MB construct the squares AMNC, MBDN, and draw the diagonals AN, MD, NB.

Imagine a straight line ab to move parallel to AB, with its ends on AC, BD, so as to pass over the rectangle ABDC; and let ab meet AN, NM, MD in n, m, d respectively. Then it is clear that all the different divisions of the line ab (=AB) will be obtained by supposing n to be *one* of the points of section (since n moves over *half* the line as ab moves from AB to CD) and *the other* to be anywhere along the line. Thus, considering the latter point of division, we may represent every possible position of it by the area of the rectangle ABDC; and we shall then have to find over what part of this area the second point of division may range, as ab moves from AB to CD, in order that it may be possible to form an *acute-angled* triangle with the three segments of the line ab.

Now no triangle of any kind could be formed with these segments unless the second point of division fell in md; for if it were in am or db, one of the three parts would be greater than half the line, and therefore greater than the sum of the other two. Again, suppose nb to be divided by the second point of section into two parts such that the sum or difference of their squares may be equal to the square on an; that is to say, suppose

$$bf^2 = an^2 + nf^2, \quad \text{or} \quad bf^2 - fn^2 = an^2 \ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$ne^2 = an^2 + be^2, \quad \text{or} \quad ne^2 - eb^2 = an^2 \ldots\ldots\ldots\ldots\ldots (\beta),$$

$$an^2 = nh^2 + hb^2, \quad \text{or} \quad nk^2 + kb^2 = an^2 \ldots\ldots\ldots\ldots\ldots (\gamma).$$

Then, if n be one of the points of section, the other must fall in fh or ke, in order that it may be possible to form an acute-angled triangle with the segments; for, according as the second point is in mf, de, or hk, the square on the right-hand, middle, or left-hand segment will be greater than the squares of the other two, and *that* segment will therefore subtend an obtuse angle in the triangle formed by the three.

Join eB; then, since $an = a\text{A} = b\text{B}$, we have from ($\beta$)

$$ne^2 = eb^2 + b\text{B}^2 = e\text{B}^2, \quad \therefore \; ne = e\text{B};$$

hence the locus of e is a rectangular hyperbola DeVe'M, of which B is a focus, AN a directrix, the point (Q suppose) where BN meets AC the centre, and the semi-axis (QV) equal to AB. It may likewise be shown that the loci of f and h (or k) are the hyperbolas Nff'M, NhkD.

Again, $bf^2 - fn^2 = an^2 = ne^2 - eb^2$, whence it follows that $be = nf$ and $de = mf$. Moreover, if $a'b'$ be that position of ab in which $a'\text{A} = a'n' = bk$, and therefore $b'n' = ak$, we shall have $b'f'^2 - f'n'^2 = a'n'^2 = bk^2 = an^2 - nk^2$; hence $b'f' = an$, and $f'm' = nm = m\text{N}$; that is to say, for any two equal abscissas Mm', Ds, the corresponding ordinates $m'f'$, $e'd'$, sk of the hyperbolic segments Nff'M, M$e'e$D, DkhN are equal, and therefore the areas of these three segments are equal. This proves, what in fact is pretty evident *à priori*, that the probability of subtending an obtuse angle is the same for each of the three parts into which the line may be divided.

Now, by a well-known formula for the area of a hyperbolic segment, we have

$$\text{MVDOM} = \text{QO} \,.\, \text{OD} - \text{QV}^2 \log_e (\text{QB} : \text{QV}) = (\tfrac{3}{8} - \tfrac{1}{2}\log 2)\,\text{AB}^2;$$

hence the area of the *curvilinear* triangle MDN is $(\frac{3}{2}\log 2 - 1)\,\text{AB}^2$; that of the *rectilinear* triangle MDN being $\frac{1}{4}\text{AB}^2$. If, therefore, we put p_1, p_2 for the respective probabilities that the three segments of the line will form an *acute* or an *obtuse* angled triangle, we shall have

$$p_1 = \frac{\text{area of curvilinear } \triangle \text{ MND}}{\text{area of rectangle ABDC}} = 3\log_e 2 - 2 = \log_e\left(\frac{8}{e^2}\right);$$

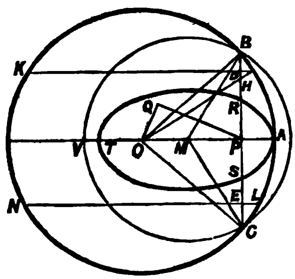
$$p_1 + p_2 = \frac{\text{area of rectilinear } \triangle \text{ MND}}{\text{area of rectangle ABDC}} = \frac{1}{4}.$$

The probability of a triangle of any kind is therefore ·25; of an acute-angled triangle ·08 nearly; and of an obtuse-angled triangle ·17 nearly.

Thus the odds in favour of an obtuse-angled triangle, as compared with an acute-angled triangle, are about 17 to 8, or a little more than 2 to 1.

1137 (Proposed by the Editor.)—Two points are taken at random within a circle, and straight lines are drawn from them to the centre and to each other; find the respective probabilities that the triangle thus formed will have, (1) its perimeter less than the diameter of the circle, or (2) the squares on its sides together less than a square inscribed in the circle, or (3) its area less than one-eighth of the same square.

Solution by the Proposer.



Let O be the centre of the given circle, P a point taken at random within the circle. Draw the radius OPA, and BPC perpendicular to OPA; in BC find two points D, E such that OP.PD = OP.PE = $\frac{1}{4}$OA^2, and draw HDK, LEN parallel to OA. With A as vertex, O, P as foci, and the middle point M of OP as centre, draw the ellipse RAST; also with M as centre and MB as radius draw the circle BCV. Then, in order to satisfy the condition (1), the third vertex Q of the triangle OPQ must fall within the ellipse RAST (= S_1 suppose); for (2) it must fall on the area BACV (= S_2); and for (3) it must fall within the zone HKNL (= S_3) unless OP < $\frac{1}{2}$OA, when Q may be any point within the given circle.

Put OA = 1, OP = x, $\angle$AOB = θ, BMO = ϕ, AOH = ψ; then

$$S_1 = \pi(1-\tfrac{1}{2}x)(1-x)^{\frac{1}{2}} = \tfrac{1}{2}\pi z(1+z^2), \text{ if } x = 1-z^2;$$

$$S_2 = (1-\tfrac{3}{4}\cos^2\theta)\,\phi + \theta - \tfrac{1}{4}\sin 2\theta; \quad S_3 = 2\psi + \sin 2\psi.$$

Now the probability that P will be between the distances x and $x+dx$ from the centre O is $2x\,dx$ or $d(x^2)$, being the part of the whole circle contained within the ring whose bounding radii are x and $x+dx$; hence, putting p_1, p_2, p_3 for the respective probabilities of (1), (2), (3), we have

$$\pi p_1 = \int_0^1 S_1\,d(x^2); \quad \pi p_2 = \int_0^1 S_2\,d(x^2); \quad \pi p_3 = \int_0^{\frac{1}{2}} \pi\,d(x^2) + \int_{\frac{1}{2}}^1 S_3\,d(x^2).$$

The integrals are readily obtained, and are as follows:—

$$\int_0^1 S_1\,d(x^2) = 2\pi\int_0^1 (z^2-z^6)\,dz = \tfrac{8}{21}\pi;$$

$$\int_0^1 S_2\,d(x^2) = \Big[\tfrac{1}{2}(S_2+\theta+\phi)\,x^2 - \tfrac{1}{8}\sin 2\theta - \tfrac{7}{12}\theta - \tfrac{3}{8}\phi\Big]_{x=0}^{x=1} = \tfrac{7}{12}\pi;$$

$$\int_{\frac{1}{2}}^1 S_3\,d(x^2) = \Big[x^2 S_3 + \psi + \cot\psi\Big]_{x=\frac{1}{2}}^{x=1} = \tfrac{3}{2}\sqrt{3} - \tfrac{1}{4}\pi.$$

The probabilities of (1), (2), (3) are thus found to be

$$p_1 = \frac{8}{21}; \quad p_2 = \frac{7}{12}; \quad p_3 = \frac{3\sqrt{3}}{2\pi}.$$

1529 (Proposed by Mr. S. BILLS.)—Find two positive rational numbers, such that (1) their *sum*, (2) their *difference*, shall be equal to the difference of their fourth powers.

Solution by MR. W. HOPPS; MATTHEW COLLINS, B.A.; MR P. W. FLOOD; *and the* PROPOSER.

1. Let $(n+1)x$ and $(n-1)x$ represent the numbers required; then

$$(n+1)x+(n-1)x=(n+1)^4x^4-(n-1)^4x^4;\ \text{whence}\ x=(4n^2+4)^{-\frac{1}{3}},$$

which is evidently rational when $n=\pm 1$. But as this value of n produces an unsatisfactory result, we may assume $n=m+1$, then $x=(4m^2+8m+8)^{-\frac{1}{3}}$, which it is necessary to make rational; and to effect this, suppose $4m^2+8m+8=(pm+2)^3$; then, assuming $12pm=8m$, or $p=\frac{2}{3}$, we find $m=\frac{9}{2}$; whence $n=\frac{11}{2}$ and $x=\frac{1}{5}$: consequently $\frac{13}{10}$ and $\frac{9}{10}$ are numbers answering the condition required in the first case of the Question. Another pair of numbers satisfying (1) is $\frac{286}{189}$ and $\frac{252}{189}$.

2. To satisfy the second case, n must be a proper fraction; hence, assuming $n=m+\frac{11}{2}$, we have $x=(4m^2+44m+125)^{-\frac{1}{3}}$, and in order to make this rational, let $4m^2+44m+125=(pm+5)^3$; then, supposing $75pm=44m$, or $p=\frac{44}{75}$, we find $m=-\frac{122625}{21296}$; whence $n=-\frac{5497}{21296}$ and $x=\frac{484}{785}$. Consequently $\frac{26793}{34540}$ and $\frac{15799}{34540}$ are numbers satisfying the second case of the Question.

1566 (Proposed by ALIQUIS.)—Prove by direct calculation that
$(\lambda\sin^2\theta+\mu\cos^2\theta)^{-1}+(\lambda\sin^2\phi+\mu\cos^2\phi)^{-1}$ is constant when
$\lambda\sin\theta\sin\phi+\mu\cos\theta\cos\phi=0.$

Solutions (I.) *by* W. SPOTTISWOODE, F.R.S.; (II.) *by the* PROPOSER.

$$\text{Let}\ \lambda\sin^2\theta+\mu\cos^2\theta=\Lambda,\ \text{and}\ \lambda\sin^2\phi+\mu\cos^2\phi=\mathrm{M},$$

$$\text{then}\ \sin^2\theta=\frac{\Lambda-\mu}{\lambda-\mu},\ \text{and}\ \sin^2\phi=\frac{\mathrm{M}-\mu}{\lambda-\mu}.$$

Substituting these values in the following equation derived from (2), viz., $\lambda^2\sin^2\theta\sin^2\phi=\mu^2\cos^2\theta\cos^2\phi$, we have $\lambda^2(\Lambda-\mu)(\mathrm{M}-\mu)=\mu^2(\Lambda-\lambda)(\mathrm{M}-\lambda)$; or, reducing, $(\lambda+\mu)\Lambda\mathrm{M}=\lambda\mu(\Lambda+\mathrm{M})$, $\therefore\ \frac{1}{\Lambda}+\frac{1}{\mathrm{M}}=\frac{1}{\lambda}+\frac{1}{\mu}=$ a *constant*.

II. Otherwise: let $\lambda=k\cos\theta\cos\phi$, $\mu=-k\sin\theta\sin\phi$; then

$$\Sigma\frac{1}{\lambda\sin^2\theta+\mu\cos^2\theta}=\left(\frac{1}{\sin\theta\cos\theta}-\frac{1}{\cos\phi\cos\phi}\right)\frac{1}{k\sin(\theta-\phi)}$$

$$=\frac{\sin 2\phi-\sin 2\theta}{2k\sin\theta\cos\theta\sin\phi\cos\phi\sin(\theta-\phi)}=\frac{-\cos(\theta-\phi)}{k\sin\theta\cos\theta\sin\phi\cos\phi}$$

$$=\frac{1}{k\cos\theta\cos\phi}-\frac{1}{k\sin\theta\sin\phi}=\frac{1}{\lambda}+\frac{1}{\mu}.$$

[NOTE.—If λ, μ be the squares of the semi-axes of an ellipse, $(\lambda\mu\Lambda^{-1})$ and $(\lambda\mu M^{-1})$ will be the squares of a pair of conjugate semi-diameters (a', b') whose inclinations to the axes are θ, ϕ; hence the foregoing Solutions contain a direct proof that $a'^2+b'^2$ is constant.—EDITOR.]

1147 (Proposed by the Hon. JAMES COCKLE, M.A., Chief Justice of Queensland.)—Show that the equation $x^5-5x^2+2=0$ can be rigorously solved.

Solution by PHILO-QUINTIO.

This equation is a particular case of the more general form $x^5-5Qx^2+2Q^{\frac{5}{3}}=0$, as given by the Proposer, in Art. 96 of his *Theory of Equations of the Fifth Degree*, in the *Phil. Magazine* for March, 1860. It can be solved after an elegant and direct manner, by means of Arts. 6, 9, 10 of the Rev. R. Harley's *Theory of Quintics*, printed in the *Quarterly Journal of Mathematics* for January, 1859; but the following process appears to be preferable for the more general form.

Assume $(x^2+ax+\beta)\left\{x^3-ax^2+(a^2-\beta)x-\frac{\beta}{a}(a^2-\beta)\right\}=$

$$x^5+\frac{1}{a}(a^4-3a^2\beta+\beta^2)x^2-\frac{\beta^2}{a}(a^2-\beta)=x^5-5Qx^2+E;$$

where β is the product of any *two* roots of the given quintic.

Hence we have the following equations of condition:—

$$a^4-3a^2\beta+5Qa+\beta^2=0\ldots\ldots\ldots\ldots(1);\quad a^2\beta^2+Ea-\beta^3=0\ldots\ldots\ldots\ldots(2);$$

$$\beta(1)+(2)\ \text{ gives }\ a^3\beta-2a\beta^2+5Q\beta+E=0\ldots\ldots\ldots\ldots(3);$$

$$a(2)-\beta(3)\ \text{ gives }\ Ea^2+a\beta^3-5Q\beta^2-E\beta=0\ldots\ldots\ldots\ldots(4);$$

$$E(2)-\beta^2(4)\ \text{ gives }\ (E^2-\beta^5)a+5Q\beta^4=0,\quad\text{whence } a=\frac{5Q\beta^4}{\beta^5-E^2}.$$

Substituting in (2) for a, we obtain finally

$$\beta^{10}-5^2Q^2\beta^7-5QE\beta^6-2E^2\beta^5+5QE^3\beta+E^4=0\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

If now we write $2Q^{\frac{5}{3}}$ for E, (5) becomes

$$\beta^{10}-5^2Q^2\beta^7-10Q^{\frac{8}{3}}\beta^6-8Q^{\frac{10}{3}}\beta^5+40Q^6\beta+16Q^{\frac{20}{3}}=0\ldots\ldots\ldots\ldots(6).$$

One root of (6) is obviously $\beta = -Q^{\frac{2}{3}}$, whence $\alpha = -Q^{\frac{1}{3}}$;

$$\therefore \quad x^5 - 5Qx^2 + 2Q^{\frac{5}{3}} = (x^2 - Q^{\frac{1}{3}}x - Q^{\frac{2}{3}})\,(x^3 + Q^{\frac{1}{3}}x^2 + 2Q^{\frac{2}{3}}x - 2Q).$$

The more general form of the quintic is consequently resolvable into factors, which become in the particular case of $Q=1$, $E=2$, the given equation

$$x^5 - 5x^2 + 2 = (x^2 - x - 1)\,(x^3 + x^2 + 2x - 2) = 0,$$

and the roots of these are those of the equation proposed.

1517 (Proposed by H. R. GREER, B.A.)—If each edge of a tetrahedron is perpendicular to the non-conterminous edge (it being observed that if two pairs of such edges be mutually perpendicular, the third pair will be so too); prove that the nine-point circles of the three triangular faces lie on a sphere; also that the nine-point circle of any triangular face, and the three points of intersection of the perpendiculars of the other three triangular faces, lie on a sphere; and find the equations of all these spheres.

I. *Solution by* MR. W. K. CLIFFORD.

The triangular equation of the nine-point circle is

$$a^2yz + b^2zx + c^2xy = 2\,(x+y+z)\,(bc\cos A\,.\,x + ca\cos B\,.\,y + ab\cos C\,.\,z)\ldots(1.)$$

Now when two opposite edges of a tetrahedron are perpendicular, a plane may be drawn through either perpendicular to the other, and will therefore contain the perpendiculars from the extremities of the first upon the second. Consequently, if a tetrahedron has *two* pairs of perpendicular edges, the perpendiculars from the vertices to opposite faces will meet in a point, and the foot of any one of them will be the intersection of perpendiculars of the face in which it lies. From this it is obvious that, if ABCD be such a tetrahedron,

$$AB\,.\,AC\cos BAC = AC\,.\,AD\cos CAD = AD\,.\,AB\cos DAB = (A), \text{ suppose.}$$

It follows at once that the nine-point circles of the four faces lie on a sphere; for the tetrahedral equation of this, the "twenty-four-point sphere," is

$$ab^2\,.\,xy + ac^2\,.\,xz + \ldots = 2\,(x+y+z+w)\left\{(A)\,x + (B)\,y + (C)\,z + (D)\,w\right\}\ldots(2.)$$

The equation to the sphere which contains the nine-point circle of the face a, and the polar centres of the other three faces, is got by changing the sign of (A) in equation (2). The equation of the self-conjugate sphere is

$$ab^2\,.\,xy + ac^2\,.\,xz + \ldots\ldots = (x+y+z+w)\left\{(A)\,x + (B)\,y + (C)\,z + (D)\,w\right\};$$

this, therefore, passes through the intersection of the circumscribed and the twenty-four-point spheres. If through the middle point of each edge of a tetrahedron a line be drawn parallel to the opposite edge, the tetrahedron will be reproduced in an inverted position. In the present case, the two tetrahedra will have the same twenty-four-point sphere, and the sphere self-conjugate to one will circumscribe the other.

II. *Solution by the* PROPOSER.

Several properties of this species of tetrahedron have been proved by Mr. Wolstenholme (*Quarterly Journal of Mathematics*, Vol. iii.); those most pertinent to our requirements in the present case being, 1st, the four perpendiculars of the tetrahedron meet in a point; 2nd, the perpendiculars from any two vertices upon any one edge meet it in the same point; 3rd, the sum of the squares of any pair of non-conterminous edges is constant, *i. e.*, $ab^2+cd^2=ac^2+bd^2=bc^2+ad^2$. An easy method of conceiving of this tetrahedron is by imagining a plane triangle (abc), through the intersection of its perpendiculars a perpendicular raised to its plane, a point d taken anywhere on this perpendicular, which point is to be the fourth vertex of the tetrahedron; and finally, the tetrahedron being constructed, we must be able to see that this construction might equally have been started from *any* of the four faces. The first of the properties stated in the Question can now be easily proved, only by observing that, if two circles have a common chord, a sphere can be described containing both. Now the nine-point circles of abc and bcd have a common chord, viz., the intercept on bc between its middle point and the foot of the perpendicular on it (from a or d); hence a sphere can be described through *any two* of the four nine-point circles; and it is evident that that described through any one pair is the same for all. For convenience, let us call this sphere the twelve-point sphere of the tetrahedron $abcd$; and let O be the intersection of the perpendiculars of the tetrahedron. Consider the tetrahedron Oabc. The twelve-point sphere of $abcd$ passes through the nine-point circle of abc, and, among other points, through the point where the perpendicular from b meets the plane Oac, this being the same point as that where the perpendiculars from a and c on bd meet it and each other; hence we see that for the tetrahedron Oabc a sphere can be described in the manner stated in the second part of the above Question; and this tetrahedron is as general a type of the particular species we are dealing with as is the original, or any other tetrahedron. This second sphere may, for a particular tetrahedron, be called the nine-point sphere. Any tetrahedron (of the present species) has one twelve-point sphere and four nine-point spheres, one of these latter corresponding to each face—the face, namely, which it intersects in the nine-point circle. Consider the five tetrahedra, $abcd$, Oabc, Obcd, Obda, Oacd. The following properties will easily be seen, some having been already proved:—

(1.) The twelve-point sphere of $abcd$ is the nine-point sphere of Oabc corresponding to the face abc, of Obcd corresponding to the face bcd, and similarly for the rest. (2.) The twelve-point sphere of Oabc is the nine-point sphere of $abcd$ corresponding to the face abc; the twelve-point sphere of Obcd is the nine-point sphere of $abcd$ corresponding to the face bcd, and so forth. (3.) The nine-point spheres of $abcd$ corresponding to the face bcd, and of Oabc corresponding to the face Obc, coincide, and similarly for all analogous combinations. (4.) The twelve-point sphere of Obcd coincides with the nine-point sphere of Oabc corresponding to the face Obc, and similarly for similar combinations.

Remembering that the equation of the nine-point circle *in plano* is

$$a\cos A\,.\,x^2+b\cos B\,.\,y^2+c\cos C\,.\,z^2-(ayz+bzx+cxy)=0;$$

it is easy to verify that the equation in quadriplanar coordinates of the twelve-point sphere for the tetrahedron $abcd$ is

$$Fx^2+Gy^2+Hz^2+Kw^2-Lxy-Myz-Nzx-Pwx-Qwy-Rwz=0,$$

where the values of the coefficients are (p_1, p_2, p_3, p_4, denoting the four perpendiculars of $abcd$) the following; viz.,

$$F=\frac{ac^2+ab^2-cb^2}{p_1^2};\quad G=\frac{bd^2+bc^2-dc^2}{p_2^2};\quad H=\frac{ca^2+cd^2-ad^2}{p_3^2};$$

$$K=\frac{bd^2+ad^2-ba^2}{p_4^2};$$

$$L=\frac{2ab^2}{p_1p_2};\quad M=\frac{2bc^2}{p_2p_3};\quad N=\frac{2ca^2}{p_3p_1};\quad P=\frac{2da^2}{p_4p_1};\quad Q=\frac{2db^2}{p_4p_2};\quad R=\frac{2dc^2}{p_4p_3}.$$

The equation of a circle passing in any plane triangle through the middle point of the base, the foot of the perpendicular on the base, and the intersection of the perpendiculars of the triangle, may be shown to be

$$a\cos A\,.\,x^2+b\cos B\,.\,y^2+2c\cos C\,.\,z^2-c\,(xy+yz\cos B+zx\cos A)=0.$$

Hence it may be verified that the equation of the nine-point sphere, corresponding to the face abc, of the tetrahedron $abcd$ is the general equation written above when the coefficients have the following values: viz.,

$$F=\frac{ac^2+ab^2-cb^2}{p_1^2};\quad G=\frac{cb^2+bd^2-cd^2}{p_2^2};\quad H=\frac{bc^2+cd^2-bd^2}{p_3^2};$$

$$K=\frac{2\,(cd^2+bd^2-bc^2)}{p_4^2};\quad L=\frac{2ab^2}{p_1p_2};\quad M=\frac{2bc^2}{p_2p_3};\quad N=\frac{2ca^2}{p_3p_1}.$$

$$P=\frac{ac^2+ad^2-cd^2}{p_4p_1};\quad Q=\frac{cb^2+bd^2-dc^2}{p_4p_2};\quad R=\frac{bc^2+cd^2-bd^2}{p_3p_4}.$$

1394 (Proposed by MATTHEW COLLINS, B.A.)—Required a *direct* proof that an ellipse and its osculating circle have a contact of the third order at the ends of its axes; also prove that the deviations of the ellipse from the circle osculating it most closely at the ends of its axes are to each other inversely as the seventh powers of the axes.

Solution by A CORRESPONDENT.

The direct proof of the second part of the theorem contains a proof of the first part: viz., the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$; and in the neighbourhood of the extremity of the major axis

$$x=a\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}=a-\frac{1}{2}\frac{ay^2}{b^2}-\frac{1}{8}\frac{ay^4}{b^4}-\text{\&c.};$$

$$\therefore\ a-x=X_e=\frac{1}{2}\frac{ay^2}{b^2}+\frac{1}{8}\frac{ay^4}{b^4};$$

if X_e be the deviation of the ellipse from the tangent, for an ordinate y. The equation of the circle of curvature is $\left(x-a+\frac{b^2}{a}\right)^2+y^2=\frac{b^4}{a^2}$; and therefore in the neighbourhood of the same point

$$x=a-\frac{b^2}{a}+\frac{b^2}{a}\left(1-\frac{a^2y^2}{b^4}\right)^{\frac{1}{2}}=a-\frac{b^2}{a}+\frac{b^2}{a}\left(1-\frac{1}{2}\frac{a^2y^2}{b^4}-\frac{1}{8}\frac{a^4y^4}{b^8}-\&c.\right);$$

$$\therefore\ a-x=X_c=\frac{1}{2}\frac{ay^2}{b^4}+\frac{1}{8}\frac{a^3y^4}{b^6},$$

if X_c be the deviation of the circle of curvature from the tangent, for the ordinate y. The difference X_c-X_e is the deviation of the ellipse from the circle of curvature, for the ordinate y; and we have

$$X_c-X_e=\frac{1}{8}\left(\frac{a^3}{b^6}-\frac{a}{b^4}\right)y^4=\frac{1}{8}\frac{a(a^2-b^2)y^4}{b^6}.$$

And since this varies as y^4, the contact is of the third order, or say it is 4-pointic. At the extremity of the minor axis we have, in like manner,

$$Y_c-Y_e=\frac{1}{8}\frac{b(b^2-a^2)x^4}{a^6},\quad\text{or}\quad Y_e-Y_c=\frac{1}{8}\frac{b(a^2-b^2)x^4}{a^6}.$$

Hence, in order to compare the deviations, writing $x=y$, we have

$$(X_c-X_e)\div(Y_e-Y_c)=\frac{a}{b^6}\div\frac{b}{a^6}=\frac{a^7}{b^7},$$

which is the relation in question.

The most simple proof of the first part of the theorem consists, however, in the remark that, if $U=0$ be the equation of an ellipse, (α,β) the coordinates of any point on the ellipse, $T=0$ the equation of the tangent at that point, then the general equation of the conic, having with the given conic a 4-pointic contact at the point (α,β) is $U+kT^2=0$. We have

$$U=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,\quad T=\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1=0;$$

and it is therefore clear that the equation can only represent a circle if either $\alpha=0$ or $\beta=0$, that is, at the extremity of one of the axes. Thus at the extremity of the major axis, writing in this case $T=x-a=0$ for the equation of the tangent, the general equation of the conic of 4-pointic contact is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+k(x-a)^2=0$, and it is clear that this will represent a circle if $k=\frac{1}{b^2}-\frac{1}{a^2}$. We in fact have identically

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\left(\frac{1}{b^2}-\frac{1}{a^2}\right)(x-a)^2=\frac{1}{b^2}\left[\left(x-a+\frac{b^2}{a}\right)^2+y^2-\frac{b^4}{a^2}\right],$$

and the equation is thus that of the circle of curvature.

Solution by MR. W. K. CLIFFORD.

1. We know that if the osculating circle at a point P meet the ellipse again at Q, PQ and the tangent at P are equally inclined to the axes. (Salmon's

Conics, 4th ed., Art. 244; Taylor's *Geometrical Conics*, p. 85.) This shows that the equation of the osculating circle at (ξ, η) may be written in either of the forms

$$\left(\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{x\xi}{a^2}+\frac{y\eta}{b^2}-1\right)\left\{\frac{\xi(x-\xi)}{a^2}-\frac{\eta(y-\eta)}{b^2}\right\} \quad\text{..............(1)},$$

$$\left(\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left\{\frac{\xi^2}{a^4}(x-\xi)^2-\frac{\eta^2}{b^4}(y-\eta)^2\right\} \quad\text{..(2)}.$$

When $\xi=0$, $\eta=\pm b$, (2) becomes

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)(y-\eta)^2=0 \quad\text{...........(3)}.$$

When $\eta=0$, $\xi=\pm a$, (2) becomes

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-\left(\frac{1}{a^2}-\frac{1}{b^2}\right)(x-\xi)^2=0 \quad\text{...........(4)}.$$

This shows that in both these cases the curves meet only where they meet the common tangent, that is, they have *four* consecutive points common; or, what is the same thing, they have contact of the *third* order. This is also seen to follow at once from the property enunciated at the outset.

2. Consider now the geometrical meaning of equation (4). Take any point P on the circle, and draw P_1QQT parallel to the major axis, meeting the ellipse in Q, Q_1, and the tangent at the vertex (A) in T. Then, if C is the centre of the ellipse and (x, y) the coordinates of P, the quantity $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ is equal to $\frac{PQ \,.\, PQ_1}{CA^2}$, and $(\xi-x)$ is PT. Hence (4) means that

$$\frac{PQ \,.\, PQ_1}{CA^2}=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)PT^2.$$

But in the limit $PQ_1=2CA$, so that we may write

$$PQ=\frac{PT^2 \,.\, CA}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right) \quad\text{..................(5)}.$$

If we make a similar construction with small letters near the extremity of the *minor* axis, we shall get from (3)

$$pq=\frac{pt^2 \,.\, CB}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right).$$

Hence $\frac{PQ}{pq}=\frac{CA}{CB}\cdot\frac{PT^2}{pt^2}$. But PT, pt are as the reciprocals of the radii of curvature, or as $\frac{CA^2}{CB}:\frac{CB^2}{CA}$.

$$\text{Therefore } \frac{PQ}{pq}=\frac{CA}{CB}\cdot\frac{CA^4}{CB^2}\cdot\frac{CA^2}{CB^4}=\frac{CA^7}{CB^7}.$$

3. We take this opportunity of setting down two other equations of the circle of curvature, which are easily deduced from (1) or (2).

The tangent at any point of an ellipse may be represented by the equation

$$lx+my=\sqrt{(l^2a^2+m^2b^2)} \quad\text{....................(6)}.$$

Let $\sqrt{(l^2a^2+m^2b^2)}=p$, then the common chord of the ellipse and the circle of curvature at the point (l, m) is represented by

$$lx-my=\frac{l^2a^2-m^2b^2}{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Hence from (2) the equation to the osculating circle is

$$p^2(l^2+m^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left\{l^2(px-la^2)^2-m^2(py-mb^2)^2\right\} \quad \ldots\ldots\ldots\ldots (8).$$

Next, at a point whose eccentric angle is ϕ, the equations to the tangent and the common chord are, respectively,

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1 \ldots\ldots (9); \quad \frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi=\cos 2\phi \ldots\ldots (10);$$

whence, putting $a^2-b^2=c^2$, we readily obtain the equation of the osculating circle in the form

$$x^2+y^2-2c^2\left(\frac{x}{a}\cos^3\phi-\frac{y}{b}\sin^3\phi\right)=a^2\sin^2\phi+b^2\cos^2\phi-c^2\cos 2\phi \ldots (11).$$

4. If we put $x=(x_1+\alpha)$, $y=(y_1+\beta)$, the equation of the tangent will still be of the form

$$l(x_1+\alpha)+m(y_1+\beta)=\sqrt{(l^2a^2+m^2b^2)} \quad \ldots\ldots\ldots\ldots\ldots (12),$$

and the line through (α, β) perpendicular to this is clearly $\dfrac{x_1}{l}=\dfrac{y_1}{m}$.

The locus of the foot of this perpendicular is therefore obtained by writing (x_1, y_1) for (l, m) in (12).

Hence, *if the equation of the tangent to any curve can be put in the form*

$$lx+my=F(l, m),$$

then the equation of the pedal with the point (α, β) *for origin is*

$$x(x-\alpha)+y(y-\beta)=F(x-\alpha, y-\beta) \quad \ldots\ldots\ldots\ldots (13).$$

For instance, the locus of the foot of the perpendicular from the centre of the ellipse on the common chord (7) is

$$(x^2+y^2)^2(a^2x^2+b^2y^2)=(a^2x^2-b^2y^2)^2 \quad \ldots\ldots\ldots\ldots\ldots (14).$$

All that has been here said may be applied to the hyperbola by changing the sign of b^2.

5. The first pedal of the cycloid may be simply obtained by the method of Art. 4. For let the circle begin to roll on the axis of x at the origin, and consider the tangent at a point corresponding to a revolution ϕ of the circle. If its equation be written $lx+my=F(l, m)$, we must have $\phi=2\cot^{-1}\left(-\dfrac{l}{m}\right)$; now the tangent passes through the point $2x=a\phi$, $y=a$, where a is the diameter of the circle; whence the equation may be written

$$lx+my=a\left(m-l\cot^{-1}\frac{l}{m}\right),$$

and we immediately get the pedal with origin (α, β); *viz.*,

$$x(x-\alpha)+y(y-\beta) = a\left\{(y-\beta)-(x-\alpha)\cot^{-1}\frac{x-\alpha}{y-\beta}\right\}.$$

Putting $\alpha=0$ and $\beta=0$, we get the pedal from the origin, viz.,

$$x^2+y^2 = a\left(y-x\cot^{-1}\frac{x}{y}\right),$$

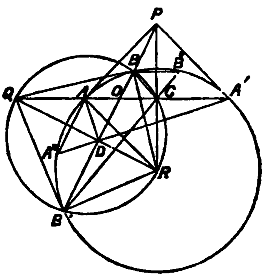
or, in polar coordinates, $r = a(\sin\theta-\theta\cos\theta)$.

1484 (Proposed by MATTHEW COLLINS, B.A.)—If two chords of a circle, AA′ and BB′, cut at O, and if OB and OB′ subtend equal angles at C, the middle of AA′; prove conversely that OA and OA′ must also subtend equal angles at D, the middle of BB′; also prove that

$$CB+CB' = DA+DA'.$$

Solution by J. R. ALLEN, *Rugby School.*

Draw the tangents AP, A′P to the circle ABA′; then the segments BO, OB′, made by AA′ on *any* chord BB′ passing through P, will subtend equal angles at the middle point C of AA′.



For take the centre R of the circle ABA′, and draw the straight line RCP: then PC . PR = PA² = PB . PB′, therefore a circle can be drawn through B, C, R, B′. Produce A′A to meet this circle in Q, and join QR. Then, because the angle QCR is a right-angle, QR is a *diameter* of the circle QBB′; and, since it passes through the centres of the two circles, QR bisects their common chord BB′ at right angles in D. Hence arc QB = QB′; ∴ ∠OCB = OCB′; that is, OB and OB′ subtend equal angles at C.

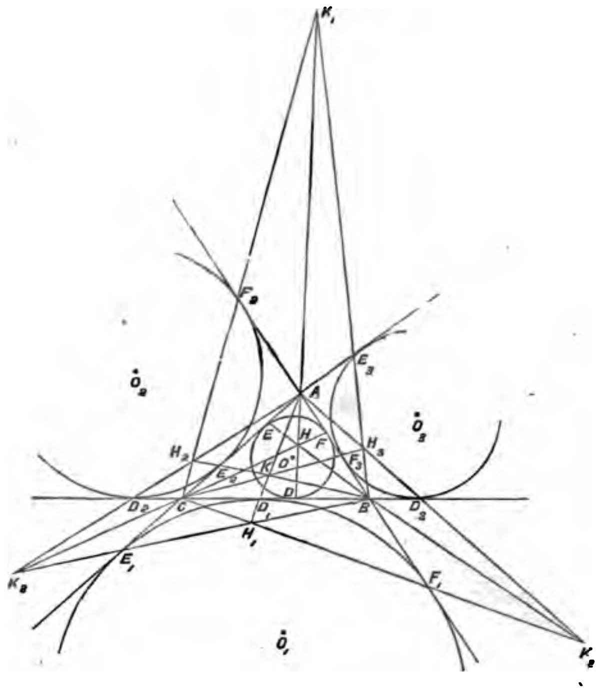
Again, since QBR, QB′R are right angles, QB and QB′ are tangents to the circle ABA′; hence, by what is proved above, the segments AO, OA′, made by BB′ on *any* chord AA′ passing through Q, will subtend equal angles at the middle point D of BB′.

Lastly, produce B′C, A′D to meet the circle ABA′ in B″, A″; then CB + CB′ = B′B″, and DA + DA′ = A′A″; also ∠B′BB″ = B′OA′ = A′AA″, hence the chords A′A″, B′B″ are equal, that is, CB + CB′ = DA + DA.

1536 (Proposed by Mr. W. GODWARD.)—Let H, H_1, H_2, H_3 be the intersections of the four triads drawn from the vertices of a plane triangle to

the points of contact of the inscribed circle and of *each* of the escribed circles, taken separately; K the intersection of the triad drawn from the vertices to the points of *internal* contact of the three escribed circles, and K_1, K_2, K_3 the intersections of the three duads drawn from the extremities of each of the sides BC, CA, AB to the points of *external* contact of two of the escribed circles with the other two sides; also let O be the centre of the inscribed circle, and O_1, O_2, O_3 the centres of the escribed circles touching BC, CA, AB respectively. It is required to prove that each of the quartads KO, K_1O_1, K_2O_2, K_3O_3 and HK, H_1K_1, H_2K_2, H_3K_3 intersects in a point, and to determine the coordinates of their points of intersection.

Solution by the PROPOSER; *and* MR. S. BILLS.



Adopting the usual diagram and notation for a triangle and its four circles of contact, we have

$AF_3 = CD_1 = s-b = s_2$, $AE_2 = BD_1 = s-c = s_3$, $BF_3 = CE_2 = s-a = s_1$;

from which we obtain the coordinates of

$$D_1 \ldots (0,\ s_2 \sin C,\ s_3 \sin B);\quad E_2 \ldots (s_1 \sin C,\ 0,\ s_3 \sin A);$$
$$F_3 \ldots (s_1 \sin B,\ s_2 \sin A,\ 0).$$

We have also the coordinates of

$$A.. \left(\frac{2\Delta}{a}, 0, 0\right); \quad B.. \left(0, \frac{2\Delta}{b}, 0\right); \quad C.. \left(0, 0, \frac{2\Delta}{c}\right).$$

From these we readily obtain the equations of

$$AD_1 .. bs_3\beta - cs_2\gamma = 0; \quad BE_2 .. as_3\alpha - cs_1\gamma = 0; \quad CF_3 .. as_2\alpha - bs_1\beta = 0.$$

As any one of these three equations may be derived from the other two, it is evident that AD_1, BE_2, CF_3 will, as is well known, intersect in a point (K); and any two of them combined with $a\alpha + b\beta + c\gamma = 2\Delta$, will give the coordinates of this point. Let K_1, K_2, K_3 be the respective intersections of CF_2, BE_3; AD_3, CF_1; BE_1, AD_2. Then we readily find the coordinates of these four K-points to be as follows; *viz.*

$$K.. \left(\frac{2\Delta s_1}{as}, \frac{2\Delta s_2}{bs}, \frac{2\Delta s_3}{cs}\right); \quad K_1.. \left(\frac{2\Delta s}{as_1}, -\frac{2\Delta s_3}{bs_1}, -\frac{2\Delta s_2}{cs_1}\right);$$

$$K_2.. \left(-\frac{2\Delta s_3}{as_2}, \frac{2\Delta s}{bs_2}, -\frac{2\Delta s_1}{cs_2}\right); \quad K_3.. \left(-\frac{2\Delta s_2}{as_3}, -\frac{2\Delta s_1}{bs_3}, \frac{2\Delta s}{cs_3}\right).$$

Also we have the coordinates of

$$O.. (r, r, r); \quad O_1.. (-r_1, r_1, r_1); \quad O_2.. (r_2, -r_2, r_2); \quad O_3.. (r_3, r_3, -r_3).$$

From these we easily obtain the equations of the following lines, after substituting the values of s, s_1, s_2, s_3 in terms of the sides a, b, c; viz.:

$$OK .. a(ac - ab - b^2 + c^2)\alpha + b(a^2 + ab - bc - c^2)\beta + c(-a^2 - ac + b^2 + bc)\gamma = 0 \ldots\ldots (1),$$

$$O_1K_1 .. a(-ac + ab - b^2 + c^2)\alpha + b(a^2 - ab - bc - c^2)\beta + c(-a^2 + ac + b^2 + bc)\gamma = 0 \ldots\ldots (2),$$

$$O_2K_2 .. a(ac + ab - b^2 + c^2)\alpha + b(a^2 - ab + bc - c^2)\beta + c(-a^2 - ac + b^2 - bc)\gamma = 0 \ldots\ldots (3),$$

$$O_3K_3 .. a(-ac - ab - b^2 + c^2)\alpha + b(a^2 + ab + bc - c^2)\beta + c(-a^2 + ac + b^2 - bc)\gamma = 0 \ldots\ldots (4).$$

The aggregate of these equations is

$$a(-b^2 + c^2)\alpha + b(a^2 - c^2)\beta + c(-a^2 + b^2)\gamma = 0 \ldots\ldots (5).$$

Now if equations (1), (2), (3), (4) be satisfied by the same values of α, β, γ, equation (5) must coexist with each of them.

But (5) will evidently be reduced to an identity by making

$$a\alpha = b\beta = c\gamma \ldots\ldots\ldots (6);$$

and it is visibly evident that the same will be the case with each of the equations (1), (2), (3), (4). It is hence manifest that each of the lines OK, O_1K_1, O_2K_2, O_3K_3, as well as (5), passes through the same point, and equations (6) at once suggest that this point is the centre of gravity of the triangle, the coordinates of which are readily found to be

$$\frac{2\Delta}{3a}, \frac{2\Delta}{3b}, \frac{2\Delta}{3c}.$$

Again, by an analogous process it will be found that each of the four triads (AD, BE, CF), (AD_1, BE_1, CF_1), (AD_2, BE_2, CF_2), (AD_3, BE_3, CF_3) intersects in a point. These four points of intersection are respectively H, H_1, H_2, H_3 in the diagram, and their coordinates have been determined to be as follows:

$$\text{H} \,..\, \frac{2\Delta s_2 s_3}{a(s_1 s_2 + s_2 s_3 + s_3 s_1)}, \quad \frac{2\Delta s_3 s_1}{b(s_1 s_2 + s_2 s_3 + s_3 s_1)}, \quad \frac{2\Delta s_1 s_2}{c(s_1 s_2 + s_2 s_3 + s_3 s_1)};$$

$$\text{H}_1 \,..\, -\frac{2\Delta s_2 s_3}{a(ss_2 + ss_3 - s_2 s_3)}, \quad \frac{2\Delta ss_2}{b(ss_2 + ss_3 - s_2 s_3)}, \quad \frac{2\Delta ss_3}{c(ss_2 + ss_3 - s_2 s_3)};$$

$$\text{H}_2 \,..\, \frac{2\Delta ss_1}{a(ss_1 + ss_3 - s_1 s_3)}, \quad -\frac{2\Delta s_1 s_3}{b(ss_1 + ss_3 - s_1 s_3)}, \quad \frac{2\Delta ss_3}{c(ss_1 + ss_3 - s_1 s_3)};$$

$$\text{H}_3 \,..\, \frac{2\Delta ss_1}{a(ss_1 + ss_2 - s_1 s_2)}, \quad \frac{2\Delta ss_2}{b(ss_1 + ss_2 - s_1 s_2)}, \quad -\frac{2\Delta s_1 s_2}{c(ss_1 + ss_2 - s_1 s_2)}.$$

These coordinates with those of K, K_1, K_2, K_3 already found, from which the common factors in each set may be omitted, will give the equations to

$$\text{HK} \,..\, a^2 s_1 (b-c)\,\alpha + b^2 s_2 (c-a)\,\beta + c^2 s_3 (a-b)\,\gamma = 0,$$
$$\text{H}_1\text{K}_1 \,..\, a^2 s\,(b-c)\,\alpha + b^2 s_1 (c+a)\,\beta - c^2 s_2 (a+b)\,\gamma = 0,$$
$$\text{H}_2\text{K}_2 \,..\, -a^2 s_3 (b+c)\,\alpha + b^2 s\,(c-a)\,\beta + c^2 s_1 (a+b)\,\gamma = 0,$$
$$\text{H}_3\text{K}_3 \,..\, a^2 s_2 (b+c)\,\alpha - b^2 s_1 (c+a)\,\beta + c^2 s\,(a-b)\,\gamma = 0.$$

Any three of these four equations will, with $a\alpha + b\beta + c\gamma = 2\Delta$, give

$$\alpha = \frac{2\Delta\,(b^2+c^2-a^2)}{a\,(a^2+b^2+c^2)} = \frac{b^2c^2 \cos \text{A}}{\text{R}\,(a^2+b^2+c^2)},$$
$$\beta = \frac{2\Delta\,(a^2-b^2+c^2)}{b\,(a^2+b^2+c^2)} = \frac{c^2a^2 \cos \text{B}}{\text{R}\,(a^2+b^2+c^2)},$$
$$\gamma = \frac{2\Delta\,(a^2+b^2-c^2)}{c\,(a^2+b^2+c^2)} = \frac{a^2b^2 \cos c}{\text{R}\,(a^2+b^2+c^2)};$$

we hence infer that HK, H_1K_1, H_2K_2, H_3K_3 intersect in a point.

Cor. 1. From the coordinates of K_1, K_2, K_3, A, B, C given above, we find the equations to

$$\text{K}_1\text{A} \,..\, bs_2\beta - cs_3\gamma = 0, \quad \text{K}_2\text{B} \,..\, as_1\alpha - cs_3\gamma = 0, \quad \text{K}_3\text{C} \,..\, as_1\alpha - bs_2\beta = 0,$$

which are the same equations as we have found for AD, BE, CF; we hence infer that K_1AHD, K_2BHE, K_3CHF are three straight lines intersecting in H, the point of intersection of AD, BE, CF.

Cor. 2. The coordinates of the points K, K_1, K_2, K_3 give the equations to

$$\text{KK}_1 \,..\, a\,(b-c)\,\alpha + b^2\beta - c^2\gamma = 0, \quad \text{K}_2\text{K}_3 \,..\, a\,(b+c)\,\alpha + b^2\beta + c^2\gamma = 0,$$
$$\text{KK}_2 \,..\, -a^2\alpha + b\,(c-a)\,\beta + c^2\gamma = 0, \quad \text{K}_3\text{K}_1 \,..\, a^2\alpha + b\,(c+a)\,\beta + c^2\gamma = 0,$$
$$\text{KK}_3 \,..\, a^2\alpha - b^2\beta + c\,(a-b)\,\gamma = 0, \quad \text{K}_1\text{K}_2 \,..\, a^2\alpha + b^2\beta + c\,(a+b)\,\gamma = 0.$$

It will be found by applying the criterion given at Art. 17, Ferrers' *Trilinear Coordinates*, to these equations, that KK_1 is perpendicular to K_2K_3, KK_2 to K_3K_1, and KK_3 to K_1K_2: it hence follows that any one of the K's is the point of intersection of the perpendiculars of the triangle whose vertices are the other three. The same property also appertains to the four centres O, O_1, O_2, O_3, as is well known.

1533 (Proposed by Professor CAYLEY.)—If on the sides of a triangle there are taken three points, one on each side; and if through the three points and the three vertices of the triangle there are drawn a cubic curve and a quartic curve, intersecting in six other points; then there exists a quintic curve passing through each of the three points, and having each of the six points for a double point.

Solution by the PROPOSER.

Let $P = 0$ be the equation of the quartic curve, $Q = 0$ the equation of the cubic curve, $M = 0$ the equation of the three sides of the triangle; then if we can find A, B, C functions of the orders 0, 1, 2 respectively, and U a function of the fifth order, such that we have identically $MU = AP^2 + BPQ + CQ^2$; we have $MU = 0$, a curve of the eighth order, having a double point at each of the points ($P = 0$, $Q = 0$), which points are the three vertices of the triangle, the three points, and the six points; but the curve $MU = 0$ is made up of the curve $M = 0$ (the three sides of the triangle, being a cubic curve having each of the vertices for a double point, and passing through each of the three points) and of a certain quintic curve $U = 0$; hence the quintic curve must pass through each of the three points, and have a double point at each of the six points; or there exists a quintic curve satisfying the conditions of the theorem.

I take $x = 0$, $y = 0$, $z = 0$ for the equations of the three sides of the triangle, and then (the constants being all of them arbitrary) writing for shortness

$$\begin{array}{lll} \xi = \;\;.\;\;\;\; by + cz, & X = \;\;.\;\;\;\; \beta y + \gamma z, & \Theta = \lambda x + \mu y + \nu z, \\ \eta = a'x \;\;\;\; . + c'z, & Y = \alpha' x \;\;\;\; . + \gamma' z, & \\ \zeta = a''x + b''y \;\;\;\; ., & Z = \alpha'' x + \beta'' y \;\;\;\; ., & \end{array}$$

I assume that the three points are given by the equations ($x = 0$, $\xi = 0$) ($y = 0$, $\eta = 0$) ($z = 0$, $\zeta = 0$), respectively. This being so, we may write

$$Q = yz\xi\delta + zx\eta\delta' + xy\zeta\delta'' + xyz\epsilon = 0, \quad -P = yz\xi X + zx\eta Y + xy\zeta Z + xyz\Theta = 0,$$

for the equations of the cubic curve and the quartic curve respectively. We have of course $M = xyz = 0$ for the equation of the three sides of the triangle, and the identity to be satisfied is $xyzU = AP^2 + BPQ + CQ^2$.

I was led to the values of A, B, C by considerations founded on the theory of curves in space. We have

$$A = \delta\delta'\delta'', \quad B = (\delta'\alpha'' + \delta''\alpha)\,\delta x + (\delta''\beta + \delta\beta'')\,\delta' y + (\delta\gamma' + \delta'\gamma)\,\delta'' z,$$

$$C = \alpha'\alpha''\delta x^2 + \beta''\beta\delta' y^2 + \gamma\gamma'\delta'' z^2 + (\gamma\beta''\delta' + \gamma'\beta\delta'')\,yz + (\alpha'\gamma\delta'' + \alpha''\gamma'\delta)\,zx$$
$$+ (\beta''\alpha'\delta + \beta\alpha''\delta')\,xy;$$

and with these values it is easy to show that the function $AP^2 + BPQ + CQ^2$ contains the factor xyz; for substituting the values of P, Q, all the terms of $AP^2 + BPQ + CQ^2$ contain explicitly the factor xyz, except the terms

$$A\,(y^2z^2\xi^2X^2 + z^2x^2\eta^2Y^2 + x^2y^2\zeta^2Z^2) - B\,(y^2z^2\xi^2X\delta + z^2x^2\eta^2Y\delta' + x^2y^2\zeta^2Z\delta'')$$
$$+ C\,(y^2z^2\xi^2\delta^2 + z^2x^2\eta^2\delta'^2 + x^2y^2\zeta^2\delta''^2);$$

and these terms will contain the factor xyz, if only the expressions $AX^2 - BX\delta + C\delta^2$, $AY^2 - BY\delta' + C\delta'^2$, $AZ^2 - BZ\delta'' + C\delta''^2$ contain respectively

the factors x, y, z. But $AX^2-BX\delta+C\delta^2$ will contain the factor x, if only the expression vanishes for $x=0$; and for $x=0$ we have

$$AX^2-BX\delta+C\delta^2=\delta\delta'\delta''(\beta y+\gamma z)^2$$
$$-[\delta'\delta''(\beta y+\gamma z)+\delta(\beta''\delta' y+\gamma'\delta'' z)]\delta(\beta y+\gamma z)+(\beta y+\gamma z)(\beta''\delta' y+\gamma'\delta'' z)\delta^2=0;$$

that is, $AX^2-BX\delta+C\delta^2$ contains the factor x; and by symmetry the other two expressions contain the factors y and z respectively. The excepted terms contain therefore the factor xyz; and there exists therefore a quintic function $U=(AP^2+BPQ+CQ^2)\div xyz$; which proves the theorem.

The values of A, B, C were obtained by considering the surface $w=\dfrac{P}{Q}$, which, as is at once seen, contains upon itself the three lines

$$\left(x=0,\ w=-\frac{X}{\delta}\right),\ \left(y=0,\ w=-\frac{Y}{\delta'}\right),\ \left(z=0,\ w=-\frac{Z}{\delta''}\right)$$

or as these equations may be written

$$\begin{array}{lllll}(x=0, & \cdot & \beta y+\gamma z & +\delta w & =0),\\ (y=0, & \alpha' x & \cdot\ +\gamma' z & +\delta' w & =0),\\ (z=0, & \alpha'' x+\beta'' y & \cdot & +\delta'' w & =0);\end{array}$$

and then seeking for the equation of the hyperboloid which passes through the three lines, this is found to be $Aw^2+Bw+C=0$, where A, B, C have the before-mentioned values.

If in the foregoing theorem the cubic is considered as a given cubic curve, and the three points as three arbitrary points on the cubic, the question then arises to find the triangle; or we have the problem proposed as Question 1607.

1537 (Proposed by Dr. SALMON, F.R.S.)—To find the condition that an asymptote of the conic given by the general equation should pass through the origin.

Solutions (I.) *by* F. D. THOMSON, M.A.; MR. H. MURPHY; *and* MR. A. RENSHAW; (II.) *by the* PROPOSER.

I. Let the equations of the conic and a line through the origin be

$$(f, g, h, l, m, n)(x, y, 1)^2=0 \ldots\ldots\ldots (1); \quad \frac{x}{\lambda}=\frac{y}{\mu}=\rho \ldots\ldots\ldots (2);$$

then for the points where (2) meets (1) we have

$$(f\lambda^2+2n\lambda\mu+g\mu^2)\rho^2+2(l\mu+m\lambda)\rho+h=0 \ldots\ldots\ldots\ldots\ldots (3);$$

and if (2) is an asymptote, (3) must have two infinite roots,

$$\therefore\ f\lambda^2+2n\lambda\mu+g\mu^2=0 \ldots\ldots\ldots\ldots (4); \quad l\mu+m\lambda=0 \ldots\ldots\ldots\ldots (5);$$

whence, eliminating $(\lambda:\mu)$, we get the required condition; viz.,

$$fl^2-2lmn+gm^2=0.$$

II. Otherwise: the polar of the origin, $mx+ly+h=0$, must clearly be parallel to one of the lines $fx^2+2nxy+gy^2=0$; whence we have the condition $fl^2-2lmn+gm^2=0$.

1542 (Proposed by Professor CAYLEY.)—If a given line meet two given conics in the points (A, B) and (A′, B′) respectively; and if (A″ B″) be the sibi-conjugate points (or foci) of the pairs (A, A′) and (B, B′), or of the pairs (A, B′) and (A′, B), then (A″, B″) lie on a conic passing through the four points of intersection of the two given conics.

Solution by ARCHER STANLEY.

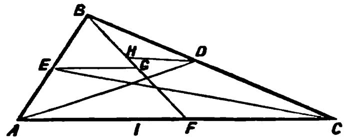
The several conics which pass through the four intersections of the two given conics form a pencil, and cut the given line in the pairs of points which constitute the involution determined by (A, B) and (A′, B′). It will suffice, therefore, to show that the pair (A″, B″) belongs to this involution. Now let (A″B″AA′) denote the anharmonic ratio $\frac{A''A}{AB''}:\frac{A''A'}{A'B''}$; then, since the points A″, B″ divide each of the segments AA′ and BB′ harmonically,

$$(A''B''AA') = (B''A''AA') = -1, \text{ and } (A''B''BB') = (B''A''BB') = -1.$$

From this, however, we at once deduce the equality (A″B″AA′) = (B″A″BB′), which, as is well known, proves that A″B″; A, B; A′, B′ are three pairs of points in involution. The same demonstration applies, of course, to the foci of the pairs A, B′ and A′, B. One of the simplest consequences of the above theorem is, that *the intersections of two similar conics lie on a circle, provided the homologous axes of these conics are at right angles to each other.* For, the conics being similar, their asymptotes enclose equal angles; but of these equal angles, the two which are of like affection (acute or obtuse), being, by hypothesis, bisected by the perpendicular homologous axes, the asymptotes of the two conics will form two pairs of right angles. Hence, considering the intersections A, B; A′, B′ of the similar and perpendicularly placed conics with the line at infinity, the segments AA′ and BB′ will each subtend a right angle at any point in the plane, and consequently the foci A″, B″ will coincide with the circular points. (Salmon's *Conics*, 4th ed., p. 307.) The conic is a circle, therefore, which passes through these points and the four intersections.

1584 (Proposed by Professor SYLVESTER.)—ABC is a system of three uniform jointed rods suspended from B; find geometrically the position of equilibrium and the directions of the pressures at A and C.

Solution by the PROPOSER.



Bisect the angles at A and C by AD, CE; make AF : FC = BD : BE; join BF; and draw DH, EG parallel to AC. Then in the position of equilibrium BF will be vertical, and HA, GC will be the directions of the pressures at A and C.

For BA, BC, AC substitute their halves at B, A; B, C; A, C respectively; then in the position of equilibrium it is seen that if BF is vertical we must have

$$\tfrac{1}{2}(BA+AC)\,AF = \tfrac{1}{2}(BC+CA)\,CF, \text{ or } AF : FC = BC+CA : BA+AC.$$

But, if the bisector of the angle B meet AC in I,

$$\frac{BC+CA}{BA+AC} = \frac{BC+CA}{AC}\cdot\frac{AC}{BA+AC} = \frac{BA}{AE}\cdot\frac{DC}{CB} = \frac{AI}{AE}\cdot\frac{DC}{CI} = \frac{BD}{BE}\ldots\ldots(a),$$

for AD, CE, BI meet in a point. This shows that in the position of equilibrium BF becomes vertical. Again, the equilibrium of A, considered as fixed between AB, AC, shows that if BF represent the weight $\tfrac{1}{2}(AB+AC)$, FA will represent the tension along AC, and AB that along AB. But the pressure on A, considered as belonging to AC, is the force which balances the tension and the weight $\tfrac{1}{2}$ AC, which latter will be measured by

$$BF\,.\,\frac{AC}{AB+AC}, \text{ or } BF\,.\,\frac{CD}{CB},$$ *i.e.* HF; hence AH represents the pressure at A considered as the extremity of AC. So too it will be seen that the pressure on the same point considered as belonging to BA is the force which balances AB, BH; that is, it is represented by HA. In like manner the pressures at C take place along and are measured by CG, GC respectively; as was to be proved.

[NOTE. The proportion (*a*) may be obtained in a slightly different manner as follows :—

$$BC+CA : BC = BA : BE, \text{ and } BA : BA+AC = BD : BC;$$

hence, compounding these two proportions, we have

$$BC+CA : BA+AC = BD : BE.$$—EDITOR.]

1152 (Proposed by EXHUMATUS.)—Three chords are drawn at random across a circle; find the relative chances of 0, 1, 2, or 3 intersections.

Solution by the EDITOR.

Taking the circumference of the circle as unity, let x and $1-x$ be the parts into which it is divided by the first chord; and let y, z be the arcs

between the ends of the second chord and one of the ends of the first; x, y, z being measured in the same direction around the circumference.

Now if a chord be drawn at random, the probability that one of its ends will be in an arc Δy, and the other in an arc Δz is $2\Delta y\,\Delta z$, whatever be the magnitude of the arcs Δy, Δz; and x^2 is the probability that both ends will be in an arc x.

Hence, when $x > y > z$, that is to say, when both ends of the second chord are in the arc x, the respective probabilities of 0, 1, 2 intersections, for all such values of x, y, z, are

$$\left.\begin{aligned} A_0 &= (1-x)^2+(x-y+z)^2+(y-z)^2, \\ A_1 &= 2\,(x-y+z)\,(1-x+y-z), \\ A_2 &= 2\,(1-x)\,(y-z). \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(\mathrm{I}).$$

When $y > x > z$, that is, when the second chord crosses the first, the respective probabilities of 1, 2, 3 intersections are

$$\left.\begin{aligned} B_1 &= (1-y)^2+(y-x)^2+(x-z)^2+z^2, \\ B_2 &= 2\,(1-y+x-z)\,(y-x+z), \\ B_3 &= 2\,(1-y)\,(x-z)+2\,(y-x)\,z. \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(\mathrm{II}).$$

Moreover, when $y > z > x$, that is, when both ends of the second chord are in the arc $1-x$, the sum of the corresponding probabilities, as $1-x$ varies continuously from 1 to 0, is clearly equal to a like sum for the arc x, since the same variations are gone through in both cases; hence we shall have to double the values derived from (I).

We may consider one end of the first chord as a fixed point from which all the others are measured; and thus Δx is the probability that the second end of the first chord will be in an arc Δx, and $2\Delta x\,\Delta y\,\Delta z$ is the probability that this end of the first chord and the two ends of the second will be simultaneously in the arcs Δx, Δy, Δz. Hence, putting p_n for the total probability of n intersections, we have

$$p_0 = \int_0^1\!\!\int_0^x\!\!\int_0^y 4A_0\,dx\,dy\,dz; \quad p_3 = \int_0^1\!\!\int_x^1\!\!\int_0^x 2B_3\,dx\,dy\,dz;$$

$$p_1 = \int_0^1\!\!\int_0^x\!\!\int_0^y 4A_1\,dx\,dy\,dz + \int_0^1\!\!\int_x^1\!\!\int_0^x 2B_1\,dx\,dy\,dz;$$

$$p_2 = \int_0^1\!\!\int_0^x\!\!\int_0^y 4A_2\,dx\,dy\,dz + \int_0^1\!\!\int_x^1\!\!\int_0^x 2B_2\,dx\,dy\,dz.$$

We may shorten the integrations by using the formulæ

$$\int_\beta^\alpha f(x)\,dx = \int_\beta^\alpha f(\alpha+\beta-x)\,dx, \quad \int_0^\alpha f(x)\,dx = \int_0^\alpha f(\alpha-x)\,dx,$$

and the results are as follows: *viz.*

$$\int_0^y 4A_0\,dz = \int_0^y 4\left\{(1-x)^2+(x-z)^2+z^2\right\}dz$$

$$= 4(1-x)^2 y + \tfrac{4}{3}\left\{x^3+y^3-(x=y)^3\right\};$$

$$\int_0^y 4A_1\,dz = \int_0^y 8(x-z)(1-x+z)\,dz = 8(1-y)\,xy+4(2x-1)y^2-\tfrac{8}{3}y^3;$$

$$\int_0^y 4A_2\,dz = \int_0^y 8(1-x)\,z\,dz = 4(1-x)\,y^2;$$

$$\int_0^x 2B_1\,dz = 2x\left\{(1-y)^2+(y-x)^2\right\}+\tfrac{4}{3}x^3;$$

$$\int_0^x 2B_2\,dz = \int_0^x 4(1-y+z)(y-z)\,dz = 4xy(1+x-y)-2x^2-\tfrac{4}{3}x^3;$$

$$\int_x^1\int_0^x 2B_3\,dy\,dz = \int_x^1\int_0^x 4(1-x)\,z\,dy\,dz = 2(1-x)^2x^2;$$

$$\int_0^x\int_0^y 4A_0\,dy\,dz = 2(1-x)^2x^2+\tfrac{4}{3}x^4;\quad \int_0^x\int_0^y 4A_1\,dy\,dz = \tfrac{8}{3}x^3-2x^4;$$

$$\int_0^x\int_0^y 4A_2\,dy\,dz = \tfrac{4}{3}(1-x)\,x^3;\quad \int_x^1\int_0^x 2B_1\,dy\,dz = \tfrac{4}{3}x(1-x)^3+\tfrac{4}{3}x^3(1-x);$$

$$\int_x^1\int_0^x 2B_2\,dy\,dz = \tfrac{2}{3}x(1-x)(1+x-x^2) = \tfrac{2}{3}x(1-2x^2+x^3);$$

$$\therefore\ p_0 = \int_0^1 \tfrac{2}{3}(3x^2-6x^3+5x^4)\,dx = \tfrac{2}{3}\left(1-\tfrac{3}{2}+1\right) = \tfrac{1}{3};$$

$$p_1 = \int_0^1 \tfrac{2}{3}(2x-6x^2+12x^3-7x^4)\,dx = \tfrac{2}{3}\left(1-2+3-\tfrac{7}{5}\right) = \tfrac{2}{5};$$

$$p_2 = \int_0^1 \tfrac{2}{3}(x-x^4)\,dx = \tfrac{2}{3}\left(\tfrac{1}{2}-\tfrac{1}{5}\right) = \tfrac{1}{5};$$

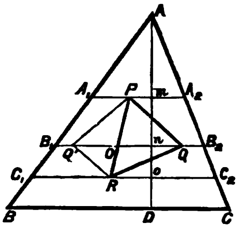
$$p_3 = \int_0^1 2(x^2-2x^3+x^4)\,dx = 2\left(\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}\right) = \tfrac{1}{15}.$$

The values of the probabilities are, therefore,

$p_0=\tfrac{5}{15},\ p_1=\tfrac{6}{15},\ p_2=\tfrac{3}{15},\ p_3=\tfrac{1}{15};$ or $p_0:p_1:p_2:p_3=5:6:3:1.$

1229 (Proposed by Mr. S. WATSON.)—Show that the average area of all the triangles that can be formed by joining three points, taken *ad libitum* upon the surface of a given triangle, is one-twelfth of the triangle.

Solution by the PROPOSER.



Let ABC be the triangle; P, Q, R any three points in it, through which draw A_1A_2, B_1B_2, C_1C_2 parallel to BC; AD a perpendicular on BC, cutting A_1A_2, B_1B_2, C_1C_2 in m, n, o; O the intersection of B_1B_2 and PR; and Q' the position of Q when on the left of PR.

Put $AD = p$, $Am = x$, $An = y$, $Ao = z$, $A_1P = x'$, $B_1Q = y'$, $C_1R = z'$, $B_1O = \delta$, $y - x = u$, $z - y = v$, $z - x = s$, and $\frac{a}{p} = \lambda$. Then

$A_1A_2 = \lambda x$, $B_1B_2 = \lambda y$, $C_1C_2 = \lambda z$; and $\frac{s}{u} = \frac{z' - x'}{\delta - x'}$, whence $\delta = \frac{uz' + vx'}{s}$.

Also, according as Q lies to the right or left of PR, the area of the triangle PQR is $\frac{1}{2}s(y' - \delta)$ or $\frac{1}{2}s(\delta - y')$: hence

$$\tfrac{1}{2}s\left\{\int_{\delta}^{\lambda y}(y'-\delta)\,dy' + \int_{0}^{\delta}(\delta - y')\,dy\right\} = \tfrac{1}{4}s\left\{(\lambda y - \delta)^2 + \delta^2\right\} \;\ldots\ldots\ldots\ldots (1)$$

expresses the sum of the areas of the triangles PQR while Q lies anywhere on B_1B_2. Restoring the value of δ, we have

$$\int_0^{\lambda x}\int_0^{\lambda z}(1)\,dx'dz' = \frac{a^4}{12p^4 s}\left(2u^2xz^3 + 3uvx^2z^2 + 2v^2x^3z\right)\ldots\ldots\ldots\ldots\ldots (2),$$

which expresses the sum of the areas when P, Q, R lie anywhere on the lines A_1A_2, B_1B_2, C_1C_2; and since these three lines can be interchanged in 6 different ways, we have, by restoring the values of u, v, s, and completing the integration of the simpler parts at each step,

$$6\int_0^p\int_0^z\int_0^y dz\,dy\,dx = -\frac{a^4p^4}{576} - \frac{a^4}{2p^4}\int_0^p\int_0^z z^4(z-y)^2\log\frac{z-y}{z}\,dz\,dy = \frac{a^4p^4}{192}.$$

This expresses the sum of the areas when P, Q, R take every position upon the triangle; and the number of positions is $\frac{1}{8}a^3p^3$; hence the average required is

$$\left(\tfrac{1}{192}a^4p^4\right) \div \left(\tfrac{1}{8}a^3p^3\right) = \tfrac{1}{24}ap = \tfrac{1}{12} \text{ of the triangle.}$$

1230 (Proposed by Mr. S. WATSON.)—Within a given triangle another is inscribed at random; what is the chance that the area of the latter does not exceed half that of the former?

Solution by the EDITOR.

Let a, b, c be the sides of the given triangle (Δ), and $\frac{1}{2}ax$, $\frac{1}{2}by$, $\frac{1}{2}cz$ the distances of the vertices of the inscribed triangle (Δ') from the middles of the sides of Δ, these distances being estimated in the same direction around the triangle; then we readily find that

$$\Delta' = \tfrac{1}{2}\Delta \text{ if } f(x, y, z) \equiv xy + yz + zx = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

Let y', y'' be values of y, and z' a value of z, such that $f(x, y', -1) = f(x, y'', 1) = f(x, y, z') = 1$; then we see by ($\alpha$) that for every value of x from -1 to 0, and for every corresponding value of y from -1 to y', there will be a value z' of z such that, for every value of z from -1 to z', the triangle Δ' will be *greater than* $\frac{1}{2}\Delta$; and this will also be the case when x, y, z are between the respective limits $(0, 1)$, $(y'', 1)$, $(z', 1)$; but for no other values of x, y, z can Δ' be greater than $\frac{1}{2}\Delta$. The values of y', y'', z' are

$$y' = \frac{x+1}{x-1}, \quad y'' = \frac{1-x}{1+x}, \quad z' = \frac{1-xy}{x+y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

Now x, y, z may, with equal probability, have any value from -1 to 1, therefore the probability that they will be simultaneously between $(x, x+\Delta x)$, $(y, y+\Delta y)$, $(z, z+\Delta z)$ is $\frac{1}{8}\Delta x\Delta y\Delta z$; hence, putting p, q for the respective probabilities that $\Delta' <$ or $> \frac{1}{2}\Delta$, so that $p + q = 1$, we have

$$8q = \int_{-1}^{0}\int_{-1}^{y'}\int_{-1}^{z'} dx\,dy\,dz + \int_{0}^{1}\int_{y''}^{1}\int_{z'}^{1} dx\,dy\,dz \quad \ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

Integrating successively, we have

$$\int_{z'}^{1} dz = 1 - \frac{1-xy}{x+y} = (1+x) - \frac{1+x^2}{x+y} = \mathrm{I}_1 \text{ (say)};$$

$$\int_{y''}^{1} \mathrm{I}_1\,dy = 2x + (1+x^2)\log\frac{1+x^2}{(1+x)^2} = \mathrm{I}_2 \text{ (say)};$$

$$\int_{0}^{1} \mathrm{I}_2\,dx = \Big[\tfrac{3}{2}(1+x)^2 + \tfrac{1}{3}(3x+x^3)\log(1+x^2)$$

$$\qquad - \tfrac{2}{3}(4+3x+x^3)\log(1+x) + \tfrac{4}{3}\tan^{-1}x\Big]_{x=0}^{x=1}$$

$$= 2 + \tfrac{1}{3}\pi - 4\log_e 2.$$

The value of the first of the two triple integrals in (γ) is the same as that of the second which has just been found; hence we have

$$q = \tfrac{1}{12}(6+\pi) - \log_e 2; \quad p = \tfrac{1}{12}(6-\pi) + \log_e 2.$$

In decimals the values of the probabilities are

$$p = \cdot 9313478, \quad q = \cdot 0686522;$$

and the first five convergents to the ratio ($q : p$) are

$$\frac{q}{p} = \tfrac{1}{13}, \tfrac{1}{14}, \tfrac{2}{27}, \tfrac{7}{95}, \tfrac{23}{312}, \text{ \&c.}$$

Hence, if a triangle be inscribed at random in a given triangle, the odds are nearly 14 to 1 against its being greater than half the given triangle.

1606. (Proposed by the EDITOR.)—Solve the following problems :—

(α) Through three given points to draw a conic whose foci shall lie in two given lines.

(β) Through four given points to draw a conic such that one of its chords of intersection with a given conic shall pass through a given point.

(γ) Through two given points to draw a circle such that its chord of intersection with a given circle shall pass through a given point.

Solution by PROFESSOR CAYLEY.

(α.) Through three given points to draw a conic whose foci shall lie in two given lines.

The focus of a conic is a point such that the lines joining it with the two circular points at infinity (say the points I, J) are tangents to the conic. Hence the question is, in a given line to find a point A, and in another given line to find a point B, such that there exists a conic touching the four lines AI, AJ, BI, BJ (where I, J are any given points) and besides passing through three given points. More generally, instead of the lines from A, B to the given points I, J, we may consider the tangents from A, B, to a given conic Θ; the question then is, in a given line to find a point A, and in another given line to find a point B, such that there exists a conic touching the tangents from A, B to a given conic Θ, and besides passing through three given points. It is rather more convenient to consider the reciprocal question—though it is to be borne in mind that for any two reciprocal questions a solution of the one question by means of coordinates (x, y, z) regarded as point coordinates is in fact a solution of the other question by means of the same coordinates (x, y, z) regarded as line coordinates. The reciprocal question is : through a given point to draw a line A, and through another given point to draw a line B, such that there exists a conic passing through the intersections of these lines with a given conic Θ, and besides touching three given lines. The given points may be taken to be $(x = 0,\ z = 0)$, $(y = 0,\ z = 0)$; this determines the line $z = 0$, but not the lines $x = 0$, $y = 0$, so that the point $(x = 0, y = 0)$ may without loss of generality be supposed to lie on the conic Θ; the equation of this conic will therefore be

$$(a, b, 0, f, g, h)\ (x, y, z)^2 = 0.$$

I take $\alpha z + \gamma x = 0$ for the equation of the line A, $\mu y + \nu z = 0$ for the equation of the line B (so that the quantities to be determined are the ratios $\alpha : \gamma$ and $\mu : \nu$); this being so, the required conic passes through the intersections of these lines with the conic Θ; its equation will therefore be

$$(a, b, 0, f, g, h)\ (x, y, z)^2 + 2\ (\alpha x + \gamma z)\ (\mu y + \nu z) = 0\ ;$$

or what is the same thing

$$(a, b, 2\nu\gamma, f + \mu\gamma, g + \nu\alpha, h + \mu\alpha)\ (x, y, z)^2 = 0\ ;$$

where α, γ, μ, ν have to be determined in such manner that this conic may touch three given lines. It is to be observed that α, γ, μ, ν, enter into the equation through the combinations $\alpha\mu$, $\alpha : \gamma$, and $\mu : \nu$, so that there are really only three disposable quantities.

The condition in order that the conic may touch a line $\xi x + \eta y + \zeta z = 0$ is

$$\left\{\begin{matrix} 2b\nu\gamma-(f+\mu\gamma)^2,\ 2a\nu\gamma-(g+\nu a)^2,\ ab-(h+\mu a)^2, \\ (g+\nu a)\ (h+\mu a)-a\ (f+\mu\gamma), \\ (h+\mu a)\ (f+\mu\gamma)-b\ (g+\nu a), \\ (f+\mu\gamma)\ (g+\nu a)-2\nu\gamma\ (h+\mu a) \end{matrix}\right\}(\xi,\ \eta,\ \zeta)^2 = 0,$$

that is, putting for shortness $C = ab-h^2$, $F = gh-af$, $G = hf-bg$, and reversing the sign of the whole expression,

$$\begin{aligned} &\left\{f^2\xi^2+g^2\eta^2-C\zeta^2-2F\eta\zeta-2G\zeta\xi-2fg\xi\eta\right\} \\ +2\mu &\left\{f\gamma\xi^2 \qquad + ha\zeta^2+(a\gamma-ga)\ \eta\zeta-(h\gamma+fa)\ \zeta\xi-g\gamma\xi\eta\right\} \\ +2\nu &\left\{-b\gamma\xi^2-(a\gamma-ga)\ \eta^2-ha\eta\zeta+ba\zeta\xi\ +\ (2h\gamma\ -\ af\ )\ \right\} \\ +\mu^2 &\left\{(\gamma\xi-a\zeta)^2\right\}\ +2\mu\nu\left\{a\eta\ (\gamma\xi-a\zeta)\right\}\ +\nu^2\left\{a^2\eta^2\right\} \qquad = 0\,; \end{aligned}$$

or what is the same thing

$$\left\{\nu a\eta+\mu\ (\gamma\xi-a\zeta)\right\}^2+2\nu\ (pa+q\gamma)+2\mu\ (ra+s\gamma)+t=0\,;$$

where $p,\ q,\ r,\ s,\ t$ are given functions of $(\xi,\ \eta,\ \zeta)$.

I write for greater convenience

$$\nu = \frac{1}{X},\ \mu = \frac{1}{Y},\ a = W,\ \gamma = Z,$$

(so that the quantities to be determined will be the ratios $X : Y : Z : W$); the foregoing equation then becomes

$$\left\{\eta\ \frac{W}{X}+\frac{1}{Y}\ (\xi Z-\zeta W)\right\}^2+\frac{2}{X}\,(pW+qZ)+\ \frac{2}{Y}\,(rW+sZ)+t = 0,$$

or what is the same thing

$$\left\{\eta YW+X\ (\xi Z-\zeta W)\right\}^2+2XY^2\ (pW+qZ)+2X^2Y\ (rW+sZ)+tX^2Y^2 = 0.$$

Hence, considering in place of the line $\xi x+\eta y+\zeta z = 0$, the three given lines $\xi_1x+\eta_1y+\zeta_1z = 0$, $\xi_2x+\eta_2y+\zeta_2z = 0$, $\xi_3x+\eta_3y+\zeta_3z = 0$ successively, we have the three equations

$$\begin{aligned} &\left\{\eta_1YW+X(\xi_1Z-\zeta_1W)\right\}^2+2XY^2(p_1W+q_1Z)+2X^2Y(r_1W+s_1Z)+t_1X^2Y^2=0, \\ &\left\{\eta_2YW+\&c, \qquad\qquad\right\}^2+\&c. \qquad\qquad\qquad\qquad =0, \\ &\left\{\eta_3YW+\&c. \qquad\qquad\right\}^2+\&c. \qquad\qquad\qquad\qquad =0; \end{aligned}$$

which, treating X, Y, Z, W as the coordinates of a point in space, are each of them the equation of a quartic surface having the line $(X = 0, Y = 0)$ for a cuspidal line. The required values of X, Y, Z, W are the coordinates of a point of intersection of the three surfaces, and these being found the equation of the conic satisfying the conditions of the question is

$$(a,\ b,\ 0,\ f,\ g,\ h)\ (x,\ y,\ z)^2+2\ (Wx+Zz)\left(\frac{y}{Y}+\frac{z}{Z}\right) = 0.$$

As to the intersection of surfaces having a common line, see Salmon's *Solid Geometry*, p. 257; but the case of a cuspidal line not having been hitherto discussed, I am not able to say now how many points of intersection

there are of the three surfaces, nor consequently what is the number of the solutions of the question in hand. It of course appears that 64 is a superior limit.

(β.) Through four given points to draw a conic such that one of its chords of intersection with a given conic shall pass through a given point.

Let the four points be given as the intersections of the conics $U=0$, $V=0$, and let $W=0$ be the equation of the given conic, (α, β, γ) the coordinates of the given point.

The equation of the required conic may be taken to be $\Theta=\lambda U+\mu V=0$, and this being so, the equation of *any* conic passing through the points of intersection of the conic $\Theta=0$ and the given conic $W=0$, will be $\lambda U+\mu V+\nu W=0$; and if ν be properly determined, viz. by the equation

$$\text{Disct.}\ (\lambda U+\mu V+\nu W)=0,$$

which it will be observed is a cubic equation in (λ, μ, ν), then $\lambda U+\mu V+\nu W=0$ will be the equation of a pair of the chords of intersection of the conics $\Theta=0$, $W=0$. The chord which passes through the given point (α, β, γ) may be taken to be one of the pair of chords; the pair of chords, regarded as a conic, then passes through the given point (α, β, γ); or if U_0, V_0, W_0 are what U, V, W become on substituting therein the values (α, β, γ) for the coordinates, we have

$$\lambda U_0+\mu V_0+\nu W_0=0,$$

which is a linear equation in (λ, μ, ν); and combining it with the before-mentioned cubic equation,

$$\text{Disct.}\ (\lambda U+\mu V+\nu W)=0,$$

the two equations give the ratios $(\lambda : \mu : \nu)$, and the equation of the required conic is $\lambda U+\mu V=0$. There are three systems of ratios $\lambda : \mu : \nu$, and consequently three conics satisfying the conditions of the Question.

Suppose that the conics $U=0$, $V=0$, $W=0$, have a common chord, then the conics $\Theta=\lambda U+\mu V=0$, $W=0$, have this common chord, say the fixed chord; and they have besides another chord of intersection, say the proper chord, which is the line joining the two points of intersection not on the fixed chord. It follows that, in the equation $\lambda U+\mu V+\nu W=0$, ν may be so determined that this equation shall represent the fixed and proper chords; the required value of ν is one of those given by the before-mentioned cubic equation, which will then have a single rational factor of the form $a\lambda+b\mu+c\nu$, and the value of ν is that obtained by means of this factor, that is, by the equation $a\lambda+b\mu+c\nu=0$.

[The value in question may, however, be found independently of the cubic equation; thus, if the three conics have the common chord $P=0$, then their equations may be taken to be $U=0$, $U+PQ=0$, $U+PR=0$; we have then $\Theta=\lambda U+\mu(U+PQ)$, and the value of ν is at once seen to be $\nu=-(\lambda+\mu)$, for then

$$\lambda U+\mu V+\nu W=\lambda U+\mu(U+PQ)-(\lambda+\mu)(U+PR)=0,$$

that is, $P\left\{\mu Q-(\lambda+\mu)R\right\}=0$, which is the conic made up of the fixed chord $P=0$ and the proper chord $\mu Q-(\lambda+\mu)R=0$.]

But returning to the equations $U=0$, $V=0$, $W=0$, the value of ν is given by the equation $a\lambda+b\mu+c\nu=0$, obtained by equating to zero the rational factor of the cubic equation. Suppose now that the *proper chord* passes through the given point (α, β, γ), then, as before, the equation $\lambda U+\mu V+\nu W$

$= 0$ is satisfied by these values of the coordinates, or we have $\lambda U^0 + \mu V^0 + \nu W_0 = 0$; which, with the before-mentioned linear equation $a\lambda + b\mu + c\nu = 0$, determines the ratios $\lambda : \mu : \nu$; and the required conic is $\lambda U + \mu V = 0$; there is, then, in the present case only one conic satisfying the conditions of the Question.

(γ.) Through two given points to draw a circle such that its chord of intersection with a given circle shall pass through a given point.

The foregoing discussion of the case of three conics having a common chord is of course directly applicable to the present question, the common chord being the line infinity; it is therefore sufficient to give the final analytical result; viz., if the given points are $y = 0$, $x = \pm 1$, and if the given circle is $x^2 + y^2 + c + 2fy + 2gx = 0$, and the point through which passes the chord is $x = a$, $y = \beta$, then the equation of the required circle is

$$x^2 + y^2 - 1 + \frac{1}{\beta}(2ga + 2f\beta + 1 + c)\,y = 0.$$

The equation of the chord of intersection is, in fact,

$$1 + c - \frac{1}{\beta}(2ga + 2f\beta + 1 + c)\,y + 2gx + 2fy = 0,$$

which is satisfied, as it should be, by $x = a$, $y = \beta$.

But the geometrical solution is even more simple. Let A, B, be the given points, C the point through which passes the chord of intersection; then, joining C, A, and taking on this line a point H such that CA . CH is equal to the square on the tangential distance of C from the given circle, it is at once seen that *any* circle through A, H is such that its chord of intersection with the given circle passes through C; hence the required circle is that drawn through the three points A, H, B.

London: Printed by C. F. Hodgson and Son, Gough Square, Fleet Street, E.C.

57

Published by C. . HODGSON & SON, Gough Square, Fleet Street.

In Twelve Numbers, interleaved with tissue paper, price 1s. each.

First Lessons in Drawing and Design;

OR,

PENCILLED COPIES AND EASY EXAMPLES.

For the use of Schools and Families, and intended as a Preparation for the Drawing Master.

BY

GEORGE CARPENTER,

PRINCIPAL DRAWING MASTER IN THE STATIONERS' COMPANY'S GRAMMAR SCHOOL, ETC. ETC.

CONTENTS.

No. 1. Straight Lines and their Combinations, &c.
No. 2. Rectilineal Figures.
No. 3. Curves.
No. 4. Outlines of Familiar Objects.
No. 5. Shaded Figures.
No. 6. Introduction to Perspective.
No. 7. Trees.
No. 8. The Human Figure.
No. 9. Animals and Rustic Figures.
No. 10. Ornament.
No. 11. Flowers.
No. 12. Maps.

The Series embraces a complete course of Elementary Drawing, consisting of appropriate and carefully graduated Copies, advancing from the simple stroke to the most difficult outline, printed in pencil-coloured ink, to be first drawn over, and then imitated. The Exercises have been so simplified as to render the art of Drawing as easy of attainment as that of Writing. As soon as children are able to write, they are also able to draw. And for the purpose of early training, in order that their ideas of Form may become correct, and the eye and hand acquire the habit of working in unison without effort, it is confidently believed that no Series of Drawing Books exists which can compare with the present.

COLLEGE OF PRECEPTORS.

Price 6d. the Set; or by post, 7d.,

THE EXAMINATION PAPERS set for the Pupils of Schools in Union with the College of Preceptors.

*** A Set of these Papers is published after each half-yearly Examination. Back Sets, from the year 1856 to 1875 inclusive, are still to be had.

COLLEGE OF SURGEONS.

Price 6d. the Set; or by post, 7d.

THE EXAMINATION PAPERS set for the Preliminary Examinations in General Knowledge of Candidates for the Diploma of Fellow and Member of the College.

*** A Set of these Papers is published after each half-yearly Examination. Back Sets, from the year 1860 to 1875 inclusive, are still to be had.

8vo, price 4d.

SCIENCE TEACHING: Remarks on the Report of the British Association on the Teaching of Science, more especially in relation to General Education. By J. ANGELL, Esq., Lecturer on Experimental Physics at the Manchester Grammar School.

8vo, price 6d.

ON TEACHING ARITHMETIC AFTER THE METHOD OF PESTALOZZI. By WALTER MCLEOD, F.R.G.S., F.C.P. Head-Master, Model Schools, Royal Military Asylum, Chelsea.

Published by C. F. HODGSON & SON, Gough Square, Fleet Street.

WORKS BY J. WHARTON, M.A.

Ninth Edition, 12mo, cloth, price 2s.; or with the Answers, 2s. 6d.,

LOGICAL ARITHMETIC; being a Text-Book for Class Teaching; and comprising a Course of Fractional and Proportional Arithmetic, an Introduction to Logarithms, and Selections from the Civil Service, College of Preceptors, and Oxford Exam. Papers.

ANSWERS TO THE QUESTIONS IN THE LOGICAL ARITHMETIC. 12mo, price 6d.

Thirteenth Edition, 12mo, cloth, price 1s.,

EXAMPLES IN ALGEBRA FOR JUNIOR CLASSES. Adapted to all Text-Books; and arranged to assist both the Tutor and the Pupil.

Third Edition, cloth, lettered, 12mo, price 3s.,

EXAMPLES IN ALGEBRA FOR SENIOR CLASSES. Containing Examples in Fractions, Surds, Equations, Progressions, &c., and Problems of a higher range.

THE KEY; containing complete Solutions to he Questions in the "Examples in Algebra for Senior Classes," to Quadratics inclusive. 12mo, cloth, price 3s. 6d.

In half-yearly Volumes, 8vo, price 6s. 6d. each. (To Subscribers, price 5s.)
Vols. I. to XXIV. are already published.

MATHEMATICAL QUESTIONS, with their SOLUTIONS, Reprinted from the EDUCATIONAL TIMES.
Edited by W. J. MILLER, B.A.

This Reprint contains, in addition to the Mathematics in each number of the *Educational Times*, about as much more original matter. Any one of the volumes can be had direct from the Publishers on the enclosure of 6s. 6d. in Postage Stamps. New Subscribers can have the back Volumes at Subscription price, 5s. 4d. including postage.

Price 6d., by post 7d. (Any Number can be had by return of post, on the enclosure of Seven postage stamps.)

THE EDUCATIONAL TIMES, and JOURNAL OF THE COLLEGE OF PRECEPTORS.

The Educational Times is published monthly; and contains, in addition to many valuable articles of a scholastic nature, Reviews, Notices, &c. &c., a most important Department for Mathematics, to which a large proportion of the leading Mathematicians of the day contribute.

8vo, Price Five Shillings.

ALGEBRA IDENTIFIED WITH GEOMETRY in Five Tracts:—1. Euclid's Conception of Ratio and Proportion;—2. "Carnot's Principle" for Limits;—3. Laws of Tensors, or the Algebra of Proportion;—4. Laws of Clinants, or the Algebra of Similar Triangles Lying on the Same Plane;—5. Stigmatic Geometry, or the Correspondence of Points in a Plane;—with one photolithographed Table of Figures. By ALEXANDER J. ELLIS, F.R.S., F.S.A.

Also, by the Same Author,

SCHOOL INDUCTIONS; or, How to Familiarize Children with the Principles of Inductive Logic. 8vo, price 1s.

READING ALOUD. A Lecture delivered before the College of Preceptors. 8vo, price 4d.